The theory of the Lévy and Lévy-type processes on $\mathbb{R}^d$ (and the corresponding non-local operators) has been intensively studied since late '90s by both probabilists and specialists in partial differential equations. This theory has real-world applications in quantum mechanics, water waves, meteorology, crystallography, biology, finance and high technology, just to name a few.

In the description of jump Lévy-type processes (and the corresponding Lévy-type operators) the fundamental object is the jumping measure $\nu(x, dy)$. Heuristically, it describes the intensity of jumping of the process from point $x \in \mathbb{R}^d$ to the point $y \in \mathbb{R}^d$. Right now, two cases are quite well explored and understood. The first case is when $\nu(x, dy)$ does not depend on x (that is when $\nu(x, dy) = \mu(dy)$). This is the case of Lévy processes (and the corresponding Lévy infinitesimal operators). The second quite well understood case is when $\nu(x, dy)$ has a density with respect to the Lebesgue measure (that is $\nu(x, dy) = q(x, y)\, dy$) and the density satisfies some regularity conditions and it is comparable to the isotropic one (that is $q(x, y) \approx q_0(|x - y|)$, typically $q(x, y) \approx |x - y|^{-d-\alpha}$, $\alpha \in (0, 2)$).

The objective of our project is to study the case when $\nu(x, dy)$ depends on x and it is singular, that is when it is not absolutely continuous with respect to the Lebesgue measure. More precisely we want to study the anisotropic Lévy-type operators $\mathcal{L}_{A,b}$ (defined below) which may be viewed as anisotropic, non-local analogs of elliptic second-order differential operators.

Let us consider a Lévy operator in $\mathbb{R}^d$: $\mathcal{L}f(x) = \lim_{\zeta \to 0^+} \int_{|z|>\zeta} [f(x + z) - f(x)]\, \nu(dz)$, where $d \geq 2$, $f \in C_b^2(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, ν is a Lévy measure on $\mathbb{R}^d$. We are mainly interested in case when the Lévy measure is anisotropic and singular, that is when it is not absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. A typical example of the operator we are interested in is $\mathcal{L}^{(s)} = -\left(-\frac{\partial^2}{\partial x_1^2}\right)^{\alpha_1/2} - \ldots - \left(-\frac{\partial^2}{\partial x_d^2}\right)^{\alpha_d/2}$, where $\alpha_i \in (0, 2)$, $i = 1, \ldots, d$.

We plan to investigate anisotropic Lévy-type operators, based on $\mathcal{L}$, which are of the following form

$$\mathcal{L}_{A,b}f(x) = \lim_{\zeta \to 0^+} \int_{|z|>\zeta} \left[f(x + zA^T(x)) - f(x)\right] \nu(dz) + b(x)\nabla f(x),$$

where $d \geq 2$, $f \in C_b^2(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, ν is as above, $A(x) = (a_{ij}(x))$ are invertible $d \times d$ matrices, $b(x) = (b_1(x), \ldots, b_d(x))$ and $a_{ij}(x)$, $b_i(x)$ are bounded and continuous functions on $\mathbb{R}^d$ with values in $\mathbb{R}$.

We plan to study properties of semigroups corresponding to these anisotropic Lévy-type operators. Due to the anisotropy of the Lévy measure ν and the effects caused by the matrix $A(\cdot)$ the operators $\mathcal{L}_{A,b}$ have some unexpected features. In particular, from our preliminary work on this theory, we learnt that for some choices of Lévy measure ν, matrices $A(\cdot)$ and $b(\cdot)$, the heat kernel $p(t, x, y)$ for $\mathcal{L}_{A,b}$ is unbounded for some $t > 0$, that is $\sup_{x,y \in \mathbb{R}^d} p(t, x, y) = \infty$ for some $t > 0$. For that reason the study of such anisotropic Lévy-type operators is difficult and the existing methods seem to be of little help.

We plan to study spectral theory for Schrödinger operators based on anisotropic Lévy-type operators, which are of the form $\mathcal{L}_{A,b} + q$. Especially interesting is the following anisotropic Lévy operator

$$\mathcal{L}^{(e)}f(x) = \sum_{i=1}^{N} \sqrt{-\frac{\partial^2}{\partial x_{3i-2}^2} - \frac{\partial^2}{\partial x_{3i-1}^2} - \frac{\partial^2}{\partial x_{3i}^2} + m^2} - Nm,$$

where $m > 0$, $N \in \mathbb{N}$. In Lieb and Seiringer's book "The Stability of Matter in Quantum Mechanics" it is a kinetic term in relativistic Schrödinger Hamiltonian for N electrons. One of aims in our research project is to investigate the spectral theory for $\mathcal{L}_{A,b}^{(e)} + q$.

We also plan to investigate boundary value theory for operators $\mathcal{L}_{A,b}$. In particular, we want to study Green functions, Poisson kernels and regularity of Dirichlet boundary value problems for these operators for bounded open sets.

We believe that our study will lead to much better understanding anisotropic, non-local operators. The ultimate goal is to build the non-local analog of the theory of elliptic second-order differential operators.