Fractional heat equation in bounded open sets

The classical heat equation was studied as early as the beginning of the nineteenth century by J. Fourier. Using the trigonometric series, later named after him, he described the temperature on a rod, given the initial temperature and the absolute zero outside the rod.

There is an equally interesting question of determining the temperature on an infinite rod (or more abstract – the real line), a plane, or a higher-dimensional Euclidean space. Such space does not have any boundary, therefore we only impose the initial temperature. This problem can be solved elegantly by integrating the initial condition multiplied by the heat kernel, also known as the Gauss–Weierstrass kernel.

The Gauss–Weierstrass kernel is a key object in the theory of stochastic processes – it describes the position of a particle which undergoes the Brownian motion. This mathematical model dates back to the beginning of the twentieth century, and we owe it, among others, to Einstein, Smoluchowski, and Wiener. Thus, if we assume, for example, that the whole heat is concentrated at a single point on the rod, then the temperature at a given moment would be exactly the probability distribution of the Brownian particle which started moving at this very point. This is arguably the most popular and elementary example of interplay between the theories of differential equations and stochastic processes.

It turns out that differential equations in bounded domains, e.g., the finite rod studied by Fourier, also can be solved using the stochastic processes. Furthermore, the probabilistic methods enable us to grasp the boundary conditions in an elegant and simple way. Namely, the solution may be obtained by studying the stochastic process upon the first moment of exit from the considered set. One may think of the particle which starts moving inside the container – we are interested in the first place of the boundary of the container that the particle hits. Then, since we deal with a random motion, we obtain a probability distribution on the said boundary. Finally, in order to find the solution, we integrate the boundary condition against this probability distribution.

In order to study the classical heat equation in the way described above, one may consider the space-time Brownian motion. It is created by adding an additional time dimension on which the process moves downwards with constant velocity.

In the last few decades the *diffusion* equations governed by operators other than the Laplacian or other second-order differential operators gained much attention. One of the most prominent examples, crucial for us, is the fractional Laplacian. The equation is then called the *fractional heat equation*. Here the underlying stochastic process is the isotropic α-stable Lévy motion. It is somewhat reminiscent to the Brownian motion, however here the trajectory of the particles may be discontinuous – the particle may perform jumps.

The aim of our studies is to precisely describe the solutions of the fractional heat equation on bounded open sets, obtained using probabilistic methods – the so called α-parabolic functions. Due to the possibility of jumps, we need the information not only on the boundary, but on the whole complement of the domain. We will establish the integral kernels which can be used to obtain the solutions. Then we will characterize the α-parabolic functions in terms of their behavior at the boundary.

There exist at least ten definitions of the fractional Laplace operator, therefore it is vital to understand the (interior) regularity of the obtained solutions, so that the most of the definitions of the operator can be applied to them. This should make our methods more appealing to the analytic-oriented researchers. This process will require a better understanding of the differentiability properties of the Dirichlet heat kernel of the fractional Laplacian, which is an interesting topic itself.

The expected outcome of our work is a better understanding of the structure of α-parabolic functions, in particular their possible wild behavior at the boundary. Moreover, as the elliptic and classical equations teach us, the mean-value property, which serves as our definition of the α-parabolic functions, is a useful tool for developing regularity results and we hope that it will find use in the research of other types of fractional differential equations.