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Studies on the structure of trusses and their applications to rings and braces

Very early in our lives we learn that there are two ways in which calculations involving multiplication and addition can be made. If we want to add 3×5 to 3×7 we can either first multiply and then add the results one to the other thus yielding $15 + 21 = 36$, or we can take out the common factor 3, add 5 to 7 and then multiply the result by 3, thus getting $3 \times 12 = 36$. Both methods lead to the same answer, and they always do, irrespective of what particular numbers we take. The mathematical principle behind this property of whole numbers is known as the *distributive law*. We learn at school that multiplication distributes over addition. More abstractly, an algebraic system which has collections of elements together with addition, subtraction and multiplication that distributes over addition (and, by inherence, also subtraction) is known as a *ring*. Rings abound in nature, they not only appear in number systems, but they allow one for efficient description for example of motion of objects or even the structure of matter. One of the immediate consequences of the distributive law is that multiplication by 0 always gives 0, and hence we cannot divide by zero.

Humans lived happily with this distributive law until around 2007, when a German mathematician Wolfgang Rump realised that to find solutions of one of the most important equations in physics, known as the Yang-Baxter equation, is the same as to describe a ring-like system which allows for all four arithmetic operations: addition, subtraction, multiplication and division, with no exclusions (hence the division by 0 is allowed and perfectly makes sense). He called such a system a *brace*. In a brace multiplication cannot distribute in the usual way over addition, otherwise the division by 0 would not be possible. The usual rule, i.e. $a\times(b+c) = a\times b+a\times c$ must be replaced by a new, corrected rule, $a\times(b+c) = a\times b-a+a\times c$. One can quickly find that in a brace division by 0 is possible, since necessarily $0 = 1$! Because of their connection with an important equation in physics and to other structures in pure mathematics, braces has become a subject of an extremely intensive study.

In late 2017, the Principal Investigator of this project observed that rings and braces can be understood as emanations of the same algebraic system. The key is to treat addition and subtraction together as a single operation on three elements $[a, b, c] = a-b+c$. Both standard and brace distributive laws can then be written in a unified way: $a\times[b, c, d] = [a\times b, a\times c, a\times d]$. The choice of c determines what kind of the distributive law between multiplication and addition one obtains (choosing $c = 0$ we obtain the standard distributive law, the brace one is obtained by choosing $c = 1$). This way of combining addition with subtraction into a single operation over which multiplication distributes provides one with a bridge between rings and braces, and in view of that the corresponding structure has been named a *truss*.

Both rings (for hundreds of years) and braces (in the last decade) have been recognised as very important and, indeed fundamental, mathematical constructs. Trusses unify them in a single system. Furthermore, trusses leave one with an option, it is not by brute force but by a conscious decision that one chooses whether to describe observed mathematical phenomena in terms of rings, braces or by some system in-between. This changes the philosophy of mathematical investigations. The aim of this project is to study properties of trusses in order to understand their nature better, and then by making appropriate choices learn more about rings and braces.