Noncommutative dynamics and Cartan C*-subalgebras: their generalizations, properties and applications

The project lies at the intersection of three fields of pure mathematics: operator algebras (specifically C*-algebras), operator theory and dynamical systems.

The theory of operator algebras originated in connection with quantum mechanics, which revolutionized the world of physics almost 100 years ago. John von Neumann, wishing to establish the appropriate mathematical framework for the new theory, began to study algebras, which are now called von Neumann algebras (or W* -algebras). Shortly after, Israel Gelfand and Mark Naimark began systematic studies of more general structures, the so-called C*-algebras. They showed that every commutative C*-algebra is an algebra of functions defined on a particular topological space. Thus in the commutative case we can recover a classical space with points. By contrast, noncommutative C*-algebras should be viewed as "noncommutative spaces" or "quantum spaces" in which we cannot "see" – recover the points as such. Thanks to the mathematical giants of the 20th century, the theory of operator algebras has reached a broader and deeper form than anyone could have predicted. The proposed project is expected to make a significant contribution to the relationship of this theory and the theory of dynamical systems and single operators.

The noncommutativity of dynamics that appears in the title of the project can be understood in two ways.

Firstly, one can associate to any classical dynamical system, where points evolve over time, a noncommutative C*-algebra - "a quantum space", which is a counterpart of the orbit space of a given dynamical system. Moreover, the celebrated Renault's theorem from 2008 says that given a classical space and the appropriate "quantum space" one can reproduce the underlying dynamical system, where the dynamical system is modelled by a (twisted) groupoid. Renault called the corresponding *C*-subalgebras Cartan* (in honor of Élie Joseph Cartan - a French mathematician, who obtained important results in the field of Lie group theory, mathematical physics, differential geometry and general group theory).

Secondly, to study "quantum dynamical systems" one needs a mathematical apparatus to describe dynamics on noncommutative C*-algebras. We will model such dynamics in the Project by the advanced apparatus of Fell bundles over groupoids and inverse semigroups. One of our main goals is to generalize Renault's Theorem to quantum setting. Hence the aim is to find conditions under which we can reconstruct the Fell bundle from an inclusion of C*-algebras. In addition, we want to describe properties of the larger C*-algebra (that models the space of orbits of the quantum dynamical system) using only dynamics given by the Fell bundle. We are interested in describing the ideals of this C*-algebra and finding criteria for pure infiniteness, nuclearity and the so-called Haagerup property. These are crucial properties for understanding and classification of C*-algebras. We will also use the developed tools to analyze the spectrum of single operators, acting either on Hilbert or on Lp-Banach spaces.