Challenges of low-dimensional dynamics in hybrid neuron models.

The project aims at applying and developing quite recent results in low-dimensional dynamics in analysis of non-trivial behaviour of some models of nerve cells activity that correspond to complex phenomena observed in real systems. It includes obtaining new theoretical results based on the questions arising during investigation of the models and then exploiting these findings in order to provide rigorous description of the model behaviour.

The application-oriented part of the project is devoted mainly to integrate-and-fire (IF or spiking) neuron models. IF models are especially interesting objects for exploration since they belong to so-called hybrid systems: they are a combination of continuous dynamical systems (given by a system of ODEs, called "subthreshold system"), accounting for excitable properties of nerve cells, with discrete events, namely the membrane voltage resetting mechanism after a certain threshold is reached, which abstracts the mechanism of spike (action potential) emission. In the last decade hybrid systems are the object of the growing attention due to multiple possible applications. Another class of models we would like to look at are map-based neuron models. These are discrete models in which evolution of the neuron membrane voltage and spike emission are captured by iterations of piecewise smooth (or piecewise continuous) maps. Although these models seem simpler when compared with hybrid or continuous neuron models and rather remain abstract from biological viewpoint, they are often created as discretization of continuous or hybrid models. Therefore, it is natural to compare repertoire of behaviours displayed by these two types of models and mechanisms generating them.

One of mathematical tools that works effectively for both these classes of models is the rotation theory. This is a branch of dynamical systems theory, whose main object is the rotation number. If the rotation number $\varrho(f, x)$ of a point x under the map f exists, then it measures the average displacement of a point on the orbit of x by f. Rotation theory dates back to Henri Poincaré, who developed it for circle homeomorphisms. However, since that time mathematicians have built up the rotation theory, among others, for circle/interval maps with discontinuities, for annulus maps and maps of the real line with almost-periodic displacement.

In hybrid neuron models interval and circle maps (often discontinuous) appear naturally in the context of the *adaptation* or *firing map*, whose iterates allow recovering of spike-trains and distinguish between qualitatively different neuron's behaviours, including phasic responses (returning to resting potential after a few spikes in response to an input), tonic firing (spiking repeatedly), *bursting* (several spikes are produced in rapid succession, followed by a quiescent phase lacking spikes) and *mixed-mode oscillations*, in which subthreshold oscillations alternate with active periods of one or more spikes. For map-based models the theory of interval and circle mappings can be applied directly to the first return map of the voltage variable which also encodes spike-patterns fired. On the other hand, at some points our research methods will be enriched e.g. by phase space methods (including non-autonomous systems) and geometric singular perturbation theory. Some of the results will be illustrated by numerical simulations.

To sum up, our aim is to investigate selected complexity measures such as rotation sets and entropy for corresponding classes of maps, develop methods of phase-space analysis for continuous autonomous and non-autonomous systems (corresponding to subthreshold dynamics in IF models) and indicate questions about the dynamical features of these systems important from the point of view of above-mentioned applications. The second goal is to apply this knowledge in analysis of spike-trains and complex oscillations in integrate-and-fire models, especially in models with varying or dynamical threshold. We will also explore biologically significant excitability properties revealed by these models and such as e.g. *post-inhibitory facilitation* or *slope-detection*, expecting to show that these properties are not limited to dynamical threshold models and also hold for models with fixed threshold but enriched with adaptation. We believe that our results and developed methods are likely to find various future applications, also beyond neurosciences.