ALGORITHMIC NULLSTELLENSATZ

The main goal of this project is to construct efficient algorithms for some graph coloring problems using the enhanced polynomial method of Alon and Tarsi. The basic idea of the method is simple: Every graph can be represented by a multivariable polynomial so that "non-roots" of this polynomial correspond to proper colorings of the graph. The celebrated Combinatorial Nullstellensatz provides then a sufficient condition for the existence of colorings (from arbitrarily prescribed lists of colors). This condition boils down to non-vanishing of a monomial with sufficiently bounded exponents, in much the same way as in the well known fact that a (one variable) polynomial cannot have more roots (over any field) than its degree. For instance, proving that a graph polynomial contains a non-vanishing monomial with at most cubic exponents implies its 4-choosability.

Our aim is to turn this purely algebraic property of graph polynomials into an efficient algorithm finding a desired coloring from given lists (or to establish impossibility of such result). This task is very hard (or impossible) in general, but we restrict our attention to selected families of graphs, for which the polynomial method appeared recently to be unexpectedly successful. These are mainly planar graphs (with some density or structural restrictions) or their relatives (1-planar graphs or biplanar graphs). For instance, it was recently proved that every planar graph can be made 4-choosable by deleting a suitable matching. Since the proof uses graph polynomials, the actual conclusion is much stronger as it implies the on-line version of the list coloring problem.

Another advantage of the polynomial method is that it makes the related enumeration problem sometimes easier. Indeed, a direct application of some algebraic tools often gives better bounds for the number of list colorings than the existing ones. This supports our hope for deriving efficient algorithms in such situations, at least in the randomized setting.

Summarizing, our intention is to conduct a systematic study of the possible algorithmic consequences and limitations of the polynomial method. It is an interesting phenomenon that the successful application of the polynomial method does not guarantee the existence of an efficient algorithm, but still there are problems where it provides enough structural insight to guide an algorithmic construction of solutions. It is unlikely that the polynomial method can always be effective, but we would like to provide some precise conditional algorithmic lower bounds. On the way, we expect to obtain progress in some purely combinatorial problems concerning list coloring, like the Earth-Moon problem or choosability of 1-planar graphs.