# Transport equation in the modern theory of partial differential equations

This project is devoted to the generalization and application of techniques known for the transport equation for new challenges arising in applied mathematics. Transport equation is probably the simplest one in the theory of partial differential equations and can be written as

$$\partial_t \mu_t + \partial_x (b(t,x)\mu_t) = 0. \tag{1}$$

Originally, the equation was solved by a fundamental observation that values at time $t = 0$ are propagated along curves called characteristics and as a consequence, solutions to (1) are constant on characteristics. Surprisingly, many real-world phenomena can be put in this setting coming from fluid dynamics, demography, cell biology or epidemiology.

In this project, we want to focus on the following objectives:

**Objective 1. Transport equations on metric spaces**. Formulation (1) requires linear structure of the underlying domain so that derivatives are well-defined. On the other hand, equivalent formulation of (1) as a propagation of $u$ on characteristics can be generalized to an arbitrary metric space. Many models studied in the current literature are of this form. As a prominent example serves traffic flows on network that can be used to assess capacity of the streets or analyze formation of traffic jams. However, to use such models, one has to be sure that they are mathematically well-posed and this is the reason why we need to develop underlying mathematical theory.

**Objective 2. Optimal control and sensitivity analysis for structured population models**. Models of the form (1) describe dynamics of populations in various areas including demography, cell biology, immunology or ecology. Therefore, in this part of the project we study version of (1) with parameter $h$:

$$\partial_t \mu_t^h + \partial_x (b(h,x)\mu_t^h) = 0, \tag{2}$$

and we are interested in analysis of functional

$$\mathcal{J}(h,t) = \int_{\mathbb{R}^d} F(x)\,\mathrm{d}\mu_t^h(x), \tag{3}$$

where $\mu_t^h$ is a measure solution to (2) while $F : \mathbb{R}^d \to \mathbb{R}$ is a given function. We note that functionals of the form (3) can be interpreted as quantities of practical importance. For example, for $F(x) = 1$ this functional provides the total number of individuals in a population. We shall also focus our attention on the nonlinear version of (2) where model functions $b$ and $c$ may depend on the solution itself.

**Objective 3. Evolutionary PDEs in roughly time-changing setting**. Many real-world phenomena are described by equations with operators rapidly changing with time. As a prototypic example may serve the flow of electrorheological fluids. These fluids are described by the system of equations:

$$\begin{gathered} \operatorname{div} \mathbf{v} = 0, \\ \partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \mathbf{S} = -\nabla p + \mathbf{g} + \nabla \mathbf{E} \cdot \mathbf{P}, \end{gathered}$$

where $\mathbf{v} = (v_1, v_2, v_3)$ denotes the velocity of the fluid, $\mathbf{S}$ is the viscous stress tensor, $\mathbf{E}$ is the electrical intensity and $\mathbf{P}$ is the polarization. When an electric field is applied, the viscous stress changes dramatically and behaves like $\mathbf{S} \sim |D(\mathbf{v})|^{r(t,x)} D(\mathbf{v})$ with some function $r(t,x)$ where $D(\mathbf{v}) = \frac{1}{2}\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right)$ is a symmetric part of $\nabla \mathbf{v}$. If changes in electric field are highly irregular, function $r(t,x)$ may not be assumed to be continuous in time $t$. Although such problems does not seem to be connected with (1), their mathematical analysis requires a nice trick known from the theory of renormalized solutions to (1) (corresponding to the case where $b$ is less regular function).