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Classification problems in topological dynamics and ergodic theory

(Description for the general public)

Dynamical systems are models of phenomena evolving in time. Their analysis helps us to understand and predict the future of processes which we describe using equations and functions. Mathematical theory of dynamical systems provides tools for analysing such models. This analysis can not only predict the behaviour of existing models, but mathematics has a unique capability of producing impossibility results. That is, mathematics can examine its limitations and show that specific outcomes will be forever beyond our reach. The most famous examples include the impossibility of

- proving the parallel postulate using the other Euclidean axioms,
- squaring the circle using ruler and compass,
- solving a general quintic polynomial by radicals.

What do these results have in common? Each comes with rules describing methods that are to be legal for a solution.

During the project, we will try to show that some classification problems arising from the theory of dynamical systems are impossible to solve using inherently countable techniques. Note that being unsolvable using inherently countable techniques is a much stronger result than algorithmic unsolvability, that is, unsolvability using inherently finite techniques. Moreover, we will try to establish a "hierarchy of difficulty" for many classification problems in dynamical systems and ergodic theory.

Studying any class of mathematical objects, we naturally try to classify them. What does it mean? Say, we want to examine the set of all humans, denoted by H, and we are interested in their height. From this perspective, two persons of the same height are indistinguishable (in the mathematical jargon we say that they belong to the same equivalence class). This leads us to a notion of an equivalence relation, that is, a division of a set (a collection of interesting objects, like H) into disjoint parts containing all members of our set we are unable to distinguish from a certain point of view (equivalence classes). Now, we can classify our equivalence classes if there is a way to associate with each class a single point in some space (think: a number), which describes the whole class. For our example, we can associate with each set of people all having the same height a single number, which is the height of every member of the class measured in centimetres (for simplicity, we round the heights to whole numbers). In this way, we have nicely classified all classes by numbers like 167, 178, or 201.

From the mathematical point of view, our association can not be arbitrary. It has to have sound structural properties, which allow us to consider the problem of finding the value associated with a given class to be computable. In the current project, we decide that the right language to rigorously express this is the language of Borel reducibility. Vaguely speaking, Borel objects on Polish spaces are defined using inductive transfinite protocols with a countable amount of initial data allowing for a sequence of better and better approximations, so that they can be "computed" up to arbitrarily small error. Thus, they are "concrete" in a way that a function summoned into existence by appeal to the axiom of choice would never be.

The goal of this project boils down to the computation of complexity of various mathematical models appearing in ergodic theory and topological dynamics.

The primary method we will use is Borel reduction - a tool for proving certain procedures are impossible. It originated in the late 1980s in the work of Friedman and Stanley (1989) and independently Harrington, Kechris, and Louveau. The idea starts with an observation: To solve A, reduce it to a problem B for which the solution is known. Turning this around: To show that solving B is impossible, start with a known impossible problem A and reduce it to B. As usual, this sounds much easier than it really is. To construct a reduction of our known impossible problem to the problem on the classification of dynamical systems requires much information about the latter.