Arithmetic and geometric approach to asymptotic invariants in algebra and algebraic geometry

abstract for the general public

The project concerns relations between arithmetic and geometric properties of varieties and studies some asymptotic invariants attached to varieties. To be more precise, a *variety* is a set of solutions of a polynomial equation. For example the equation

$$x^2 + y^2 = z^2 \tag{1}$$

can be interpreted as the equation of a circle with radius z in the Euclidean plane with coordinates x and y. In the form the equation resembles the Pythagorean Theorem. It is well known that there exist right angled triangles whose all sides are integers. The most famous is that with sides $(3, 4, 5)$ but there are others, for example $(5, 12, 13)$. It is natural to wonder how many such (integral) solutions of equation (1) exist. It turns out that there are infinitely many and their structure is well understood.

Bringing a small change to equation (1) leads to equation

$$x^3 + y^3 = z^3 \tag{2}$$

or more generally to

$$x^n + y^n = z^n \tag{3}$$

for $n \geqslant 3$. How about positive integral solutions now? This question is known as Fermat's Last Theorem. It turns out that there are no solutions to (3) for any $n \geqslant 3$. This has been proved relatively recently by Andrew Wiles.

One can start from other point of view. Given a finite set of (integral) points in the plane, we ask what is the least degree of a curve passing through all of them at least once? More generally, what is it if we want to visit every point 2 times? Or n times for $n \geqslant 1$? Letting n grow in this question is viewed as the asymptotic approach mentioned in the title. One expects that there is a uniform answer depending on the number of points, which is valid for all n sufficiently large (there could be some exceptions for small values of n).

The specific goal of the project is to solve 2 conjectures which remain open for over 30 years. Unfortunately their statement is too complicated to quote here. However tackling them would lead to significant progress in the theory and thus would result in extending charts of knowledge available to the mankind.