Schrödinger and Klein-Gordon operators

Schrödinger operators and *Klein-Gordon operators* are both 2nd order differential operators motivated by *Quantum Mechanics* and *Quantum Field theory.* Schrödinger operators are defined as the sum of the *Laplacian* and multiplication by the *potential.* If we replace a flat space by a curved *Riemannian manifold*, we need to replace the Laplacian by the so-called *Laplace-Beltrami operator.* In Quantum Mechanics, Schrödinger operators are used to describe quantum systems: they generate their evolution and determine energy levels.

According to Einstein's *Special Relativity*, our spacetime can be modelled by the *Minkowski space.* Propagation of fields on the Minkowski space is described by the *Klein-Gordon equation.* The Klein-Gordon operator, which determines the Klein-Gordon equation, is similar to the Schrödinger operator, however it has the opposite sign in front of one of the second derivatives, which radically changes its properties. Following again Einstein, one can go one step further and suppose that the spacetime is curved—in technical terms, it is described by a *pseudo-Riemannian manifold* with the *Lorentzian signature.* It is straightforward to generalize the Klein-Gordon operator to curved spacetimes.

The project will be devoted to the study Schrödinger and Klein-Gordon operators. It will be divided into two tasks.

In Task 1 we plan to investigate various classes of 1-dimensional Schrödinger operators that can be solved in terms of special functions, e.g. the famous *Gauss hypergeometric function.* The operators will be organized in *holomorphic functions* depending on complex parameters.

The main motivation for this task comes from Quantum Mechanics, where *selfadjoint* Schrödinger operators play a prominent role. We include in our project complex potentials corresponding to nonselfadjoint Schrödinger operators, whose physical motivation is less obvious. However, this seemingly non-physical regime is also useful for physical applications, since it allows us to obtain bounds on the physical case by powerful methods of complex analysis.

The main topic of Task 2 will be the study of the Klein-Gordon operator on curved spacetimes. Here the main motivation comes from Quantum Field Theory. In order to compute *scattering amplitudes* one needs to know the so-called *Feynman propagator*, which formally can be viewed as one of the inverses of the Klein-Gordon operator. A naive approach to the computation of scattering amplitudes leads to divergent expressions. One can repair this problem by the procedure called *renormalization.* In order to fix the renormalization prescription one needs to know details of the asymptotics of the Feynman propagator close to the diagonal. We expect that our approach, based on the *Weyl quantization*, will lead to an efficient method of renormalization of quantum fields on curved spacetimes.

Our research will follow the standards of rigor typical for mathematics. At the same time we expect that a large part of our results will shed light on topics close to quantum physics.