Lévy processes and non-local Schrödinger operators

We will investigate non-local Schrödinger operators of the form $H = -L + V$, where L is the generator of a symmetric Lévy process with jumps in $\mathbb{R}^d$, $d \geq 1$, and V is a sufficiently regular deterministic or random potential.

Main motivations for this project come from the quantum physics. The classical Schrödinger equation is one of the most important equations in quantum mechanics. It describes how the quantum state of a physical system changes in time. The basic object of interest in this theory is the Schrödinger operator, which is the Hamiltonian of the system. For a single particle moving in an electrostatic field with potential V, its Hamiltonian, in the position representation, takes the form $H = H_0 + V$, where $H_0 = -\Delta$ (here Δ is the classical Laplacian), and V is understood as the multiplication operator. Stationary states of the system are in one-to-one correspondence with solutions to the eigenproblem $H\Phi = E\Phi$, i.e., with eigenfunctions and eigenvalues of the Schrödinger operator H (so-called energy states and energy levels of the system).

The operator H_0 corresponds to the quantization of the kinetic energy. According to the theory of special relativity, the choice $H_0 = -\Delta$ is appropriate at low energies only, and it leads to erroneous results at high energies. Such relativistic corrections to the classical theory can be realized through mathematical models based on *non-local Schrödinger operators*. In this case one takes $H_0 = -L$, where L is the generator of a *symmetric Lévy process with jumps*. Important examples in this class are Schrödinger operators with *fractional kinetic terms* of the form

$$H_0 = (-\Delta + m^{2/\alpha})^{\alpha/2} - m \qquad \text{and} \qquad H_0 = (-\Delta)^{\alpha/2}, \qquad \alpha \in (0,2),\ m > 0,$$

replacing the classical Laplacian. For $\alpha = 1$ these operators are called *quasi-relativistic* and *ultra-relativistic* Hamiltonians, respectively. They were both used in the study of the problem of *stability and instability of relativistic matter*. Eigenfunctions and eigenvalues of non-local Schrödinger operators can be effectively investigated via the Schrödinger semigroups $\left\{e^{-tH} : t \geq 0\right\}$, and these semigroups can be studied by probabilistic methods based on the Feynman–Kac formula. This is due to a very clear probabilistic background lying behind these operators – Lévy processes with jumps. For this very reason, mathematical models based on non-local Schrödinger operators can be interpreted in a probabilistic way, as it was done by E. Nelson in the classical stochastic quantum mechanics.

Over the last 30 years, the theory of non-local Schrödinger operators underwent rapid development, stimulated by physics problems. Nevertheless, many important questions concerning their basic spectral and analytic properties remain open as of today.

This research project offers a framework suitable for investigating a large class of non-local Schrödinger operators by a combination of probabilistic and analytic methods. Our approach will be based on the Feynman–Kac formula and will use some modern tools from probabilistic potential theory of the underlying Lévy processes. We will include kinetic terms of the form $H_0 = \phi(-\Delta)$, where ϕ is an operator monotone function, which covers the operators highlighted above and many others. We will study models with both attracting and repulsive (confining) potentials as well as operators perturbed by $\mathbb{Z}^d$-ergodic alloy-type random fields which play the crucial role in the theory of disordered media (Anderson model).

Our research problems focus on fundamental properties of evolution semigroups and basic spectral objects corresponding to non-local Schrödinger operators. In particular, we will study small-time asymptotics of Schrödinger semigroups, L^p-regularity of intrinsic Schrödinger semigroups, and the behaviour of the related ground state-transformed jump processes which are a counterpart of the classical $P(\phi)_1$-process known from the Nelson stochastic quantum mechanics. We will also find sharp estimates for isolated eigenvalues and describe the transitions in the decay rates of the corresponding eigenfunctions. In the case of random alloy-type potentials, we will study the Lifshitz tails of the integrated density of states for long-range single site potentials, and describe the large time asymptotics of the probabilistic solutions to the corresponding parabolic Anderson problems. We will also develop the theory for discrete non-local Schrödinger operators.

These problems belong to the intersection areas of several fields of mathematics and physics such as probability and stochastic processes, theory of PDE's, functional analysis and mathematical physics. We expect that our results will be of interest to a wide group of experts working in all of these areas.