Colorings, cliques, and independent sets in graph classes

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Graphs are abstract structures whose role is to model interactions between pairs of objects. Such objects are represented by vertices of the graph, and the interactions are represented by edges that connect pairs of the vertices. The simplicity and generality of these structures have made graph theory one of the most important and flourishing branches of finite mathematics. Furthermore, since many practical problems can be modeled by graphs, development of efficient graph algorithms constitutes a very important area of research in computer science.

A set of vertices in a graph is called an *independent set* if no two of its members are connected by an edge. A *coloring* of a graph is a partition of its vertices into independent sets, which are then called *colors*. Two classical combinatorial optimization problems are to find a largest possible independent set and to color the graph with the least possible number of colors. They are particularly important for practical applications of graphs to modeling conflicts in resource management (such as job scheduling and frequency assignment in cellular networks), where independent sets and colorings represent conflict-free subsystems and partitions into such subsystems.

For general graphs, the above-mentioned problems are too complex—we are unable and, most likely, will never be able to solve them by efficient algorithms, even approximately. In order to sidestep this difficulty, instead of general graphs, we deal with classes of graphs satisfying some structural constraints. In the present project, we consider two natural ways of defining such constraints: by specifying a *geometric representation* which the graphs considered must admit or a *forbidden structure* which the graphs considered cannot contain.

The first group of proposed research tasks concerns the relation between the minimum number of colors sufficient to color a graph and the size of a largest *clique* in that graph, that is, a set of vertices mutually connected by edges. In a clique, every vertex must have a distinct color, and therefore the graph requires at least as many colors as is the size of a largest clique. However, there are graphs with no cliques on more than two vertices which still require arbitrarily many colors. Classes of graphs in which the required number of colors is bounded in terms of the size of a largest clique are called *χ-bounded*. Basic questions related to χ-boundedness concern the following issues:

- Which classes of graphs are χ-bounded?
- In classes of graphs that are χ-bounded, how fast can the required number of colors grow with the size of a largest clique (in particular—is the growth polynomial)?
- In classes of graphs that are not χ-bounded, how fast can the required number of colors grow with the number of vertices when the maximum size of a clique is fixed?

In recent years, huge progress has been made in the study of χ-boundedness (with significant contribution of the PI), which has led to full or partial solutions to numerous classical problems from the 1980s, those concerning the first above-mentioned question in particular, and has provided new tools for dealing with χ-bounded classes of graphs. In the present project, we focus primarily on the other two questions and also on a related problem—verifying whether the respective graph classes satisfy the so-called Erdős-Hajnal property, that is, whether the graphs of these classes contain cliques or independent sets of polynomial size (this property is weaker than polynomial χ-boundedness). The famous Erdős-Hajnal conjecture asserts that this property holds in every class of graphs with at least one forbidden structure, and we hope our research will contribute to the resolution of this conjecture as well.

The second group of proposed research tasks concerns efficient algorithms finding independent sets in the respective classes of graphs. We look for exact algorithms, which always find a largest independent set and whose measure of efficiency is the time spent on computation, as well as for so-called approximation algorithms, which find an approximately largest set and whose primary measure of efficiency is the accuracy of approximation. This area has also recently encountered significant progress, but many important problems, such as the existence of a polynomial-time algorithm in classes of graphs with a forbidden path, remain open. We hope that our study of the problems of the first group will bring new insights into our understanding of the considered graph classes and will open the way to developing desired algorithms or to obtaining theoretical justification for the lack of such algorithms—so-called conditional lower bounds on computational complexity.