"Combinatorics and Geometry of Matroids and Polytopes"

Matroids and lattice polytopes are fundamental structures in both combinatorial optimization and combinatorial algebraic geometry. Hence they are a part of mathematics where the interaction of algebra and geometry with combinatorics is particularly strong and significant. This research area is currently very active, and our project lives at this interface. We distinguish six groups of research topics.

The toric ideal of a matroid. Toric varieties are a class of algebraic varieties that on the one hand capture many varieties seen in applications and on the other hand are more amenable to combinatorial techniques than general algebraic varieties. Indeed, the geometry of a toric variety is fully determined by the combinatorics of its associated fan. When an algebraic variety is constructed using only combinatorial data, one expects to have a combinatorial description of its defining equations and other algebraic structures. An attempt to achieve this description often leads to surprisingly deep combinatorial questions. White's conjecture about the toric ideal of a matroid is an example. It asserts that the toric ideal associated to a matroid is generated by quadratic binomials corresponding to symmetric exchanges. White's conjecture resisted numerous attempts since its formulation in 1980. Our ultimate goal would be to resolve White's conjecture, as well as stronger questions of Herzog and Hibi about Koszulness and existence of a quadratic Gröbner basis. We will investigate properties of a resolution of the toric ideal of a matroid, in particular analyze properties of the associated Betti table.

Matroid base exchange properties. This part of the project studies the most fundamental concept in matroid theory, namely base exchange properties. Surprisingly, these are not yet fully understood. Basis graphs of matroids, that is graphs capturing information about single element exchanges between bases, have been studied intensively in the 1960s and 1970s. They are quite well-described, e.g. their local characterization is known. However, many fundamental questions (like connectivity, diameter) about their natural subgraphs remain fully open. Closely related is the cyclic ordering conjecture, which provides a necessary and sufficient condition for a matroid to possess a cyclic ordering.

The matroid base polytope. There is a well-understood hierarchy of properties of convex lattice polytopes that have implications for the corresponding toric varieties. For instance, if a polytope has a regular unimodular quadratic triangulation, then its toric ideal has a quadratic Gröbner basis. The existence of such a triangulation is the strongest in this hierarchy. It is an open problem for matroid base polytopes which are known to posses the weakest property in the hierarchy – normality. Quite recently matroid base polytopes were proved to possess a slightly stronger property – the integral Carathéodory property, as conjectured by Cunningham. Our aim is to discover combinatorial properties of the class of matroid base polytopes and its subclasses.

$(1,-1)$**-polytopes associated to combinatorial objects.** It is now a well-established topic in algebraic combinatorics to study a combinatorial object through its associated lattice polytope, and further through the corresponding toric variety. Exactly the same way as for a matroid we study the matroid base polytope and the toric ideal of a matroid. Another well-known example is the order polytope associated to a partially ordered set. In most cases such polytopes are subpolytopes of a unite cube, in particular they do not have interior points. We would like to go beyond these constructions, replacing the characteristic functions used so far by signed characteristic functions. As a result, we obtain a polytope that has the origin in its interior. Thus, except all classical questions, we will study when we obtain reflexive polytopes and what are their properties.

Reciprocity for matroids. The chromatic polynomial of a graph is a basic algebraic invariant. When we evaluate the chromatic polynomial of a graph on -1 we obtain the number of its acyclic edge orientations. This is an example of a 'reciprocity theorem'. Our aim is to find one for matroids.

Almost covers. In many point-line (point-surface) geometries, to cover all the points except one, more lines are needed than to cover all points. One of the instances of this phenomenon is when we want to cover using affine hyperplanes all vertices of the n-dimensional cube but one – almost covers. A theorem of Alon and Füredi guarantees that in this case n hyperplanes are needed. While, all vertices can be covered with only 2 hyperplanes. Proofs of this kind of results almost always require algebraic tools, like Alon's Combinatorial Nullstellensatz. One of versions of the problem studied in the project are almost k-covers of the n-dimensional cube, but we will also search for new examples.