# Tensor ranks and rank additivity property

This project concerns the research on tensors. A *tensor* is a multidimensional array. More formally, an $d$-way or $d$-order tensor is an element of the tensor product of $d$ vector spaces. We can think on it as a point in a $d$-dimensional space. A first order tensor is a vector, a second-order is a matrix, An example of third order tensor is the Rubic cube, where in every of the small cube (and in the middle of a Rubic cube) we place a number. For simplicity we will not going further than the case when $d = 3$.

In 1927, Hitchcock proposed the idea of the polyadic form of a tensor, i.e., expressing a tensor as the sum of a finite number of simple tensors. The *simple tensor* (of order $d$) is a tensor product of $d$ vectors.

The *rank of a tensor* $p$ is the minimal number $R(p)$ of simple tensors, such that $p$ can be expressed as a linear combination of them. Thus $R(p) = 0$ if and only if $p = 0$, and $R(p) = 1$ if and only if $p$ is a simple tensor. In general, the higher the rank is, the more complicated $p$ "tends" to be.

The definition of tensor rank is an exact analogue to the definition of matrix rank, but the properties of matrix and tensor ranks are different. For instance there is no universal algorithm to calculate tensor rank. Our main interest is the additivity of the tensor rank. If we are given two tensors in independent vector spaces, do they have the rank additivity property, i.e., is the rank of their sum equal to the sum of their ranks? A positive answer for this question was widely known as Strassen's Conjecture , until it was disproved by Y. Shitov in 2017.

In an informal language, the property says that given two independent tensors in distinct tensor spaces, it is not easier to work with both of them at the same time than to work with them separately. In a case when these are matrix multiplication tensors, it says that there is no faster way to multiply both pairs of matrices than to perform these two multiplication separately. Unfortunately Shitov did not give the explicit counterexample. He just show it exist.

We would like to characterize families of pairs of tensors with rank additivity property and find explicit examples of pairs without the property.

In applications in the other brunches of science and technology frequently we need only approximation of a tensor rank. The name which denote how many simple tensors we need to obtain ideal approximation (as close as we want to) is the *tensor border rank*. We can ask similar question as for the tensor rank. Is border rank additive? We already know it is not true in every case. We even know the explicit example when it is not true, but would like to find smallest one.

Another problem is to check if the tensor has a given rank. We does not have universal method for doing that. We would like to find one which is better than current methods.