Multigraded Hilbert Schemes

One of the basic objects of study in algebraic geometry are *projective algebraic varieties*. The *projective n-space* $\mathbb{P}^n$ consists of $n+1$-tuples of complex numbers, where two such tuples $(a_0,\dots,a_n)$ and $(b_0,\dots,b_n)$ are identified if there exists a non-zero complex number λ such that $a_i = \lambda b_i$ for $i = 0,\dots,n$. A fixed finite number of homogeneous polynomials in $n+1$ variables define a projective subvariety of $\mathbb{P}^n$ - the locus of points on which all these polynomials vanish. One of the simplest examples is a finite collection of points of $\mathbb{P}^n$. For fixed positive integers n and r, there is a projective scheme (a generalization of a projective variety) called *Hilbert scheme* and denoted $Hilb^r_n$ that parametrizes all r-tuples of points in $\mathbb{P}^n$ and their generalizations (called *zero-dimensional subschemes of length r*), obtained by considering points with multiplicity. This means that points of $Hilb^r_{n-1}$ corresponds in a natural way to zero-dimensional subschemes of $\mathbb{P}^n$ of length r. Let $S = \mathbb{C}[x_0,\dots,x_n]$ be the polynomial ring in $n+1$-variables. To each geometric object - r-tuple of points there are corresponding algebraic objects - *ideals* in the polynomial ring S. Moreover for each geometric object R there is a unique way to choose one algebraic object that is in certain sense optimal. This ideal is called the *saturated* ideal of R.

Haiman and Sturmfels described multigraded Hilbert schemes. Their construction contains $Hilb^r_{n-1}$ as a special case. This project is concerned with two problems related to multigraded Hilbert schemes. The first is studying a particular special case that is closely related to the Hilbert scheme described above. For fixed positive integers r,n one may consider a multigraded Hilbert scheme $H^{f_{n,r}}_S$. Its points correspond to ideals in S that describe zero-dimensional subschemes of $\mathbb{P}^n$ of length r. However, unlike in the case of the Hilbert scheme $Hilb^r_{n-1}$, the scheme $H^{f_{n,r}}_S$ parametrizes not geometric objects - subschemes of $\mathbb{P}^n$ but their algebraic descriptions - ideals of S. Some points of $H^{f_{n,r}}_S$ correspond to saturated ideals, i.e. the canonical choice of algebraic description of the associated geometric object. We will denote the locus of these points $U_{n,r} \subseteq H^{f_{n,r}}_S$. The aim of the project is to classify all ideals corresponding to points of $H^{f_{n,r}}_S$ that are in some sense close to a point from $U_{n,r}$. This problem is related to the border rank of polynomials. Therefore, an effective classification of those ideals should give new results in the area of ranks of tensors.

The second task of the project is studying the connectivity of multigraded Hilbert schemes. Intuitively, a space is connected if it is in one piece. The Hilbert scheme $Hilb^r_{n-1}$ is always connected. There is an example of a non-connected multigraded Hilbert scheme. One of the aims of the project is to find conditions, under which a multigraded Hilbert scheme is connected. Moreover, the above mentioned example is quite large and therefore it would be nice to find a simpler example of a non-connected multigraded Hilbert scheme.