ERGODIC THEORY: JOININGS, EFFECTIVE DISJOINTNESS AND APPLICATIONS - ABSTRACT

1. Scientific aims: The project is oriented to raise and solve interesting purely dynamical problems around various disjointness and rigidity phenomena that result from the celebrated Sarnak's conjecture on Möbius orthogonality (from 2010) about randomness of the Möbius function $\boldsymbol{\mu}$, namely:

(S) $\lim_{N\to\infty}(1/N)\sum_{n\leq N} f(T^n x)\boldsymbol{\mu}(n) = 0$

for all $x \in X$ and all "observables" $f \in C(X)$ on a compact metric space X on which dynamics is represented by a **zero entropy** homeomorphism $T : X \to X$. (S) captures our intuition that a random sequence (here represented by $\boldsymbol{\mu}$) cannot correlate with a "low complexity observables". The Möbius function $\boldsymbol{\mu} : \mathbb{N} \to \{-1, 0, 1\}$ is one of the most important functions in number theory: it is multiplicative and its randomness is expressed by cancelations of $+1$s and -1s in $\lim_{N\to\infty}(1/N)\sum_{n\leq N}\boldsymbol{\mu}(n) = 0$ which is equivalent to the Prime Number Theorem, while $\sum_{n\leq N}\boldsymbol{\mu}(n) = \mathrm{O}(N^{1/2+\varepsilon})$, for each $\varepsilon > 0$, is already equivalent to the Riemann Hypothesis.

While (S) looks as a problem of topological dynamics, it is, in fact, a problem of ergodic theory, and to see it is enough to consider $\boldsymbol{\mu}$ as a point in $\{-1, 0, 1\}^{\mathbb{N}}$ (equipped the left shift S) and write the LHS in (S) as the integral of $f \otimes \theta$ ($\theta(y) = y_0$) defined on $X \times \{-1, 0, 1\}^{\mathbb{N}}$ against the "empiric" measure $\frac{1}{N}\sum_{n\leq N}\delta_{(T\times S)^n(x,\boldsymbol{\mu})}$. As the set of measures is compact, any weak limit κ of these empiric measures yields a **joining** between an invariant measure for T and a certain measure for the subshift $X_{\boldsymbol{\mu}}$ determined by $\boldsymbol{\mu}$. In view of the numerical Daboussi-Delange-Kátai-Bourgain-Sarnak-Ziegler (DDKBSZ) criterion (see below), given a dynamical system (X, T), the orthogonality of observables with a fixed bounded multiplicative function is strongly related to the classical (Furstenberg) disjointness of different prime powers of a measure-preserving automorphism. This is a bridge between (analytic) number theory and ergodic theory which raises some fundamental questions in ergodic theory itself: better understanding of disjointness in case of non-ergodic systems, a possibility of obtaining numerical sequences that distinguish between deterministic and non-deterministic systems, trying to find the best possible ergodic counterpart of the DDKBSZ, or proving a Prime (or Semiprime) Number Theorem for zero entropy dynamical systems to mention just important directions of research. Some more detailed questions especially concerning so called parabolic dynamics and disjointness and rigidity phenomena follow.

The Möbius system $X_{\boldsymbol{\mu}}$ has its natural factor given by $\boldsymbol{\mu}^2$ (the characteristic function of the set of square-free numbers) which, as noticed by Sarnak, has positive entropy. A natural playground to examine the subshift $X_{\boldsymbol{\mu}^2}$ is the theory of $\mathscr{B}$-free systems which also is an important direction of research in the project with a few important open problems to solve.

2. Influence on ergodic theory. Novelty: The most novel parts of the project concern: a Semiprime Number Theorem in dynamics (which is considered for the first time), dynamical consequences of short interval convergence in dynamics, while the whole concept that an internal disjointness holds for zero entropy systems would point out a completely new and fundamental difference between determinism and chaos in the theory of dynamical systems.

3. Methodology: Central role is played by the DDKBSZ numerical criterion: *if a bounded sequence* (f_n) *satisfies* $\lim_{N\to\infty}(1/N)\sum_{n\leq N} f_{pn}\overline{f}_{qn} = 0$ *for all different, sufficiently large primes* $p \neq q$, *then* $\lim_{N\to\infty}(1/N)\sum_{n\leq N} f_n\boldsymbol{u}(n) = 0$ *for each bounded, multiplicative* $\boldsymbol{u}$, which when applicable in the dynamical context (we use this criterion with $f_n := f(T^n x)$) is "morally" responsible for (internal) Furstenberg disjointness of prime powers of a measure-preserving automorphism. When, additionally, the ergodic sums $\frac{1}{N}\sum_{n\leq N} f(T^{pn}x)\overline{f(T^{qn}x)} \to 0$ with a controllable speed (this is what we call an effective disjointness) then we may expect not only Möbius orthogonality but also a Prime (or Semiprime) Number Theorem in dynamics. Moreover, the short interval convergence for multiplicative functions has a dynamical counterpart and raises questions about dynamical consequences of this type of "disjointness" of ergodic sums. In general, we use methods from modern ergodic theory: the theory of joinings, entropy and complexity, spectral theory together with tools from convex analysis, representation theory, the theory of stationary processes and topological dynamics.