Abstract for the general public

The *Hilbert transform* of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by means of the singular integral

$$Hf(t) = \frac{1}{\pi}\,\mathrm{pv}\!\int_{-\infty}^{\infty} \frac{f(s)}{t-s}\,ds.$$

It is one of the fundamental objects in harmonic analysis, and it plays an important role in various other areas of mathematics, as well as in signal processing. It was already proved by David Hilbert in 1905 that Hf is 'of the same order of magnitude' as f. More formally, f and Hf have equal L^2 norms:

$$\int_{-\infty}^{\infty} |Hf(t)|^2 dt = \int_{-\infty}^{\infty} |f(t)|^2 dt.$$

In 1928, Marcel Riesz proved a related bound for the L^p norm, where $p \in (1, \infty)$: there is a constant A_p such that

$$\int_{-\infty}^{\infty} |Hf(t)|^p dt \leqslant (A_p)^p \int_{-\infty}^{\infty} |f(t)|^p dt.$$

The optimal value of A_p was not known until the work of Stylianos Pichorides from 1972, where he proved that the above inequality holds with $A_p = \max\{\tan \frac{\pi}{2p}, \cot \frac{\pi}{2p}\}$, but fails if A_p is less than this number.

The function $f(t)$ corresponds to a *signal* in continuous time. In practical applications, however, one often works with signals in discrete time, that is, with doubly infinite sequences (a_n), with integer indices. For this reason, it is interesting to study the discrete analogues of the Hilbert transform. There is, however, more than one natural way to discretize this operator. The naive definition is

$$\mathcal{H}a_n = \frac{1}{\pi} \sum_{k \neq 0} \frac{a_k}{n-k}\,.$$

A somewhat more useful discretisation is given by the Riesz–Titchmarsch transform

$$\mathcal{H}_{\mathrm{RT}} a_n = \frac{2}{\pi} \sum_{k\text{ odd}} \frac{a_k}{n-k}\,.$$

Both operators, $\mathcal{H}$ and $\mathcal{H}_{\mathrm{RT}}$, have been studied already by Marcel Riesz and Edward Charles 'Ted' Titchmarsch in 1920s. These two researchers proved that the inequalities

$$\sum_n |\mathcal{H}a_n|^p \leqslant (B_p)^p \sum_n |a_n|^p,$$
$$\sum_n |\mathcal{H}_{\mathrm{RT}} a_n|^p \leqslant (C_p)^p \sum_n |a_n|^p$$

hold with some constants B_p, C_p, and they conjectured that the optimal values of B_p and C_p are equal to the optimal value of A_p for the continuous transform H. In fact, E.C. Titchmarsch gave a proof of this claim, which was soon found to contain an error. Despite the efforts of many prominent mathematicians, the question whether the optimal values of constants A_p, B_p, C_p are equal remained open for almost a century, save for a special case when $p = 2^n$ or $p = 2^n/(2^n - 1)$ for some $n = 1, 2, \ldots$

It is relatively clear that $C_p \geqslant B_p \geqslant A_p$. In 2017, Rodrigo Bañuelos from Purdue University and the principal investigator managed to prove that indeed the optimal values of A_p and B_p are equal. This result was published in the prestigious *Duke Mathematical Journal* in 2019. The main novelty of the proof is a link between the *discrete* Hilbert transform and a certain operation in *continuous* variable: a *martingale transform* driven by an appropriate two-dimensional diffusion process.

The main objective of this project is two-fold. First, the scope of applications of the method discussed above will be explored in order to describe the class of related transforms that can be handled in a similar way. Then, we will search for different methods which might allow us to eventually prove that the optimal value of C_p is also equal to that of A_p and B_p.