PROBABILISTIC ASPECTS OF THE TAKENS EMBEDDING THEOREM

The proposed project concerns probabilistic aspects of the *Takens embedding theorem* - classical result in the theory of *dynamical systems*. This theory originated from studying mathematical aspects of the Newtonian gravity. It was realized that for systems of three or more bodies (e.g. the solar system), their time evolution under gravitational forces (given as a solution of a certain differential equation) does not admit a simple formula representation. Starting from Poincaré in early twentieth century, qualitative methods of analysis were introduced to study such systems. Over time, these methods gave rise to an abstract mathematical field of dynamical systems, however keeping strong ties with applications (in particular to mathematical physics). It was also realized early that important tools in analysis of dynamical systems are stochastic methods, known today as the *ergodic theory* (originating in works of Boltzmann on statistical physics).

The Takens embedding theorem, which is the cornerstone of our project, deals with the problem of reconstructing a dynamical system from a sequence of measurements performed via a given observable. Consider an experimentalist observing a physical system modelled by a discrete-time dynamical system, i.e. iterations of a given transformation on a given phase space (potentially embedded in a high-dimensional space). It often happens that, instead of an actual sequence of k consecutive states of the system, the observer's access is limited to the values of k *measurements* of a real-valued *observable* taken in these states. By an observable we understand a function which assigns a real number to every point of the phase space (e.g. temperature). Therefore, it is natural to ask, to what extent the original system can be reconstructed from this sequence of measurements. This question has inspired a number of mathematical results, known as *Takens-type delay embedding theorems*, stating that the reconstruction of the underlying system is possible for certain observables, as long as the number k of known measurements is large enough. The first result of this type was a celebrated theorem of Floris Takens (1980), stating that if the phase space is a manifold (geometrically "nice" set), then the initial state of the system can be uniquely recovered if the the number of measurements is larger than *twice the dimension* of the phase space. Later this result was extended to deal also with sets of more complicated geometrical structure. This is especially useful for chaotic dynamical systems, which often admit a fractal attractor (a set to which every point converges under the dynamics) and one is interested in understanding the dynamics on the attractor itself. Moreover, these results allow one to consider the accessible plot of k-dimensional vectors of measurements as a reliable representation of the underlying physical system in the k-dimensional space. Takens-type theorems are considered as theoretical results which serve as a justification of the validity of actual procedures used by experimentalists.

In the proposed project we would like to explore *probabilistic* aspects of the Takens theorem. We consider a scenario in which the accessible initial states of the system are coming from a given probability distribution and one is interested in reconstructing the system subject to this distribution. In other words, we want the sequence of measurements to recover the initial state not for every possible state, but for a *typical* state generated by a given random generator. This assumption is often justified from the point of view of applications. Taking this point of view, Barański, Gutman and Śpiewak provided recently a probabilistic version of Takens theorem. The key conclusion is that in such a setting, it suffices to take the number of measurements to be larger than the dimension of the space, hence their number can be reduced twice with respect to the non-probabilistic case. Actually, the possibility of reducing twice the number of required measurements in the probabilistic setting was conjectured by Shroer, Sauer, Ott and Yorke (*Phys. Rev. Lett.* 80, 1998). However, the above result does not settle their conjectures, as Shroer, Sauer, Ott and Yorke consider a different notion of dimension, require different properties of the reconstruction process and conjecture a more precise scaling law for the error probability. The first goal of the project is to understand connections between these problems and work towards the conjectures and their generalizations.

The second goal of the project is to give further extensions of the probabilistic Takens theorem. Firstly, we aim at understanding how large can be the set of exceptional points and observables (i.e. the ones for which the theorem does not hold) in terms of its dimension. Secondly, we will work on obtaining additional regularity properties of the reconstruction process. We will consider also a probabilistic version of the Takens theorem for finite-dimensional subsets of infinite-dimensional spaces. This could be applied e.g. to finite-dimensional attractors for the solutions of two-dimensional Navier-Stokes equations.

The expected result of the project is developing a theory of probabilistic Takens theorems, with results being possibly applicable to actually used procedures of reconstructing a given system from a sequence of measurements. From mathematical perspective, the expected results will contribute to a better understanding of probabilistic aspects of the theory of embeddings for dynamical systems.