Small gaps between almost primes

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The concept of primality plays a central role in analytic number theory. Nowadays, it is even more important than it was in previous centuries due to the applications in modern cryptography. There are many conjectures focused around primes like the so-called twin prime conjecture or the more general Hardy−Littlewood k-tuple conjecture. The latter implies for instance that there exist infinitely many n such that each element from the set $\{n, n+2, n+6, n+8\}$ (which is a special case of *an admissible k-tuple* for $k = 4$) is prime. This conjecture is far beyond the reach of current methods, although we are able to reformulate it a bit and then it becomes amenable to our attacks. For example Maynard proved that every admissible triple (like for example $\{n, n+2, n+6\}$) contains numbers which has only 7 prime factors in total for infinitely many n. There also exist similar results also for integers k greater than 3.

Our first goal is to find the lowest possible ϱ_k such that every admissible k-tuple contains at most ϱ_k prime factors in total for infinitely many n.

Results described above are usually proven via strong tools enabling us to tightly estimate the number of primes in arithmetic progressions. In the context of sieve theory, the classic example is Bombieri−Vinogradov theorem. Therefore our minor second goal is to study the relationship between so called Elliott–Halberstam conjecture (EH) and its broader version (GEH) which generalizes it also for the almost primes – the numbers with a low number of prime factors. The author believes that EH actually implies GEH in some specific cases. Such a result would enhance certain currently known results like these concerning small gaps between primes.

Studying the Elliott–Halberstam conjecture and almost prime k-tuples is important for the same reason – we wish to get a knowledge about the structure of 'true' primes. In the first case it is very straightforward since the Elliott–Halberstam conjecture is all about distribution of primes in arithmetic progressions; in the second case the point is that almost primes resemble primes very much. They can be seen as a training ground for methods created in order to study the structure hidden behind the primes. This way, we can get a better insight into the limitations of our techniques. Perhaps, certain issues, like *the parity problem* for instance, will be better understood in the future and finally resolved. The field is very much alive and its future looks bright for many new tools were invented in recent years and many new are probably yet to be found.