Optimal concentration inequalities

Description for the general public

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It was the ancient Greeks who asked the following question: which shape of a fixed perimeter, drawn on the plane, has the biggest area? They knew that the answer is the circle, but they were not yet equipped with a formalism for proving it rigorously. The first proof has been given only in the 19th century. The goals of this project are closely related to the above problem and they concern the so-called concentration of measure phenomenon. The simplest case when this phenomenon occurs is the case of a unit Euclidean sphere in an n-dimensional Euclidean space, that is the set of points within distance one from a given point. Let us consider a subset A of the sphere, whose surface area is equal half of the surface are of the sphere (an example of such a set in a half-sphere) and look at the set A_+ of points on the sphere of distance from A not exceeding 10^{-6}. One might think that A_+ does not differ too much from the set A, as the sphere has diameter 2. It turns out however that for large n the set A_+ occupies more than 99% of its surface. It is a hard to imagine high-dimensional effect.

The present project concerns concentration of measure for convex sets in Euclidean spaces. Instead of a sphere we consider arbitrary convex set K and its subset A whose volume is equal to half of the volume of K. It is fundamental in geometry of convex sets to ask questions about the volume of the set A_t of point within distance t from the set A. This problem has important applications concerning algorithms for computing volumes of convex sets, but it also has theoretical significance. In particular, it is worthwhile from theoretical point of view to deriving precise estimates on the volume of A_t. We shall deal with the so-called optimal concentration inequalities, where the definition of the set A_t is slightly modified (the way we measure the distance between points depends on the set K under consideration). Roughly speaking, the bounds we consider are in some sense the strongest possible concentration bounds that a given measure can satisfy. Our goal is to obtain results as close to these optimal bounds as possible.

The problems under consideration are closely related to central conjectures of convex geometry. We believe that our investigation will contribute to better understanding of certain high-dimensional effects.