Weak and measure–valued solutions for equations of mathematical physics

Many important physical phenomena are believed to be reasonably well described by systems of nonlinear partial differential equations (*conservation laws* or *balance laws*) – for instance the motion of fluids or behaviour of elastic materials. However, despite the practical importance of these problems and steadfast efforts of mathematicians, the general global existence theory for many such systems remain wide open.

The fundamental equations of fluid mechanics date a long time back – 250 years to Euler and 150 years to Navier and Stokes – yet, in three dimensions global existence of solutions is known only in a weakened sense and no uniqueness can be guaranteed. Worse still, recent results of De Lellis and Székelyhidi show that for a large class of initial data one can construct an infinite number of solutions to the incompressible Euler system – from physical point of view these solutions are nonsensical. For instance, they can create and annihilate energy in almost any prescribed way. It was long believed that one should be able to guarantee uniqueness of solutions by selecting *the* physical solution, which satisfies a certain *admissibility condition* (e.g an energy inequality). However, the examples produced by the method of *convex integration* show this to be a hope in vain.

The deficiency of global well–posedness results applies also to compressible fluids, nonlinear elastodynamcis, and general systems of conservation laws. It is classically known that such systems can develop singularities (shocks), even for smooth initial data – this leads to a loss of information, which is very difficult to quantify. Not much is known of compressible flows after shock formation. This is what a famous mathematician Peter D. Lax called a "scientific scandal and a challenge".

Scarcity of well–posedness results as well as counterexamples to uniqueness even in the class of admissible solutions, motivates the need to seek new solution paradigms for nonlinear systems of conservation laws. One approach is to consider *measure–valued* solutions – i.e. to allow the solution to be a weaker object (a measure) than an integrable function. This notion of solution has been shown in many particular situations to be vast enough to support global existence theory.

In this project we will study some important physical properties of weak and measure–valued solutions, both for particular systems (compressible Euler system, polyconvex elasticity, Euler–Korteweg system) – where one can take advantage of specific additional structure – and for general systems of conservation and balance laws. In particular, we will focus on the issue of energy (or more generally entropy) conservation for weak solutions, and the property of measure-valued–strong uniqueness. Studies on the former will focus on investigating minimal regularity requirements for weak solutions so that they conserve total physical energy. The latter is a conditional uniqueness property, stating that (usually highly non-unique) measure–valued solutions become unique once a strong solution with the same initial data is known to exist.