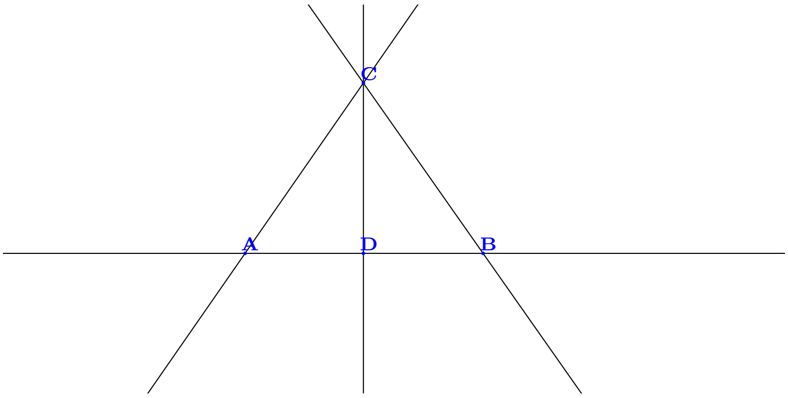
Singular curves on algebraic surfaces
Description for the general public

In this project, we will consider configuration of curves, but let us start with the simplest example, i.e. lines in the real plane. These lines may intersect each other in certain points, at some of them only two lines meet, at other places three and more. The set of intersection points is called the singular locus, and such a set of lines is called an arrangement or a configuration of lines.

One can compute the self-crossing count c of a given configuration by counting number of lines through each intersection point, squaring it and adding. If we consider the arrangement of four lines with four intersection points A, B, C, D, where A, D, B are double points and C is the triple point, the number c is equal to $2^2 + 2^2 + 2^2 + 3^2 = 21$.



For us a slightly different number is crucial, i.e. we subtract the self-crossing number from the square of the number of all lines d. This number $d^2 - c$ may be positive, zero or negative. In our example of four lines and four points, we have $4^2 - 21 = -5$. It is quite easy to find configurations of lines for which the number $d^2 - c$ is very large, very negative, or even zero, so an interesting question that we can ask is what can happen if we divide $d^2 - c$ by the number of singular points n. In our example, one has $(d^2 - c)/n = -5/4$. It is an interesting task to find a line configuration $\mathcal{L}$ for which the number $H(\mathcal{L}) = \frac{d^2-c}{n}$ is the lowest possible. It is not known what is the minimum value of $H(\mathcal{L})$, but it is proven that $H(\cdot)$ must be greater than -4 – for details please consult [1, Theorem 3.3]. This kind of questions lies in the current research interests of algebraic geometers. In the project, we are not going to restrict our attention only to line configurations since we would like to find possible bounds for values $H(\mathcal{C})$ for curve configurations $\mathcal{C}$ in algebraic surfaces. For example, such a configuration can be the union of a circle and one line intersecting each other in two points. Since number c and n can be understood in the same way as in the case of line configurations, thus the only one thing that should be done is to find a general definition of the number d – it is the so-called degree of $\mathcal{C}$. In that example this number is equal to 3 – 1 is counted for the line and 2 is counted for the circle. Now we can compute the number $H(\mathcal{C})$, which is equal to $\Big(3^2 - (2^2 + 2^2)\Big)/2 = 0.5$. Just like for line configurations, it is not know how negative $H(\mathcal{C})$ could be for conic-line configurations $\mathcal{C}$. This is actually the key point of our interest.

The problem of finding the most negative value of $H(\mathcal{C})$ addressed in the previous part is closely related to the so-called bounded negativity conjecture (see [2] for details), which is a long standing and intriguing problem in the theory of curves in algebraic surfaces. In this project we would like to investigate this conjecture for a larger class of surfaces and curve configurations. Moreover, we want to study curve configurations in the context of open problems in combinatorial algebra, namely whether certain algerbaic properties of objects associated with these configurations are determined by combinatorial features. This topic is related with the famous Terao conjecture for hyperplane arrangements in projective spaces [3].

References

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[2] Th. Bauer, B. Harbourne, A. L. Knutsen, A. Küronya, S. Müller–Stach, X. Roulleau, and T. Szemberg, Negative curves on algebraic surfaces, *Duke Math. J.* **162**: 1877 – 1894 (2013).

[3] H. Terao, *The exponents of a free hypersurface.* Singularities, Part 2 (Arcata, Calif., 1981), pp. 561 – 566, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, 1983.