Normality, determinism, generic points and various notions of disjointness for actions of amenable semigroups

Popular Science Summary of the Project

A real number x is called *simply normal* if in the decimal expansion of x, each of the digits 0, 1, ..., 9 appears equally often. For a number to be just *normal*, it has to satisfy a similar condition concerning the appearances of all finite blocks of digits. One of the consequences of Bernoulli's Law of Large Numbers is that *almost all* real numbers in $[0, 1]$ are normal (i.e., the event of randomly selecting a non-normal number has probability zero). However, the numbers we deal with most of the time (the rational numbers) are never normal, while normality of popular irrational numbers, such as e, π or $\sqrt{2}$, is unknown and this problem is extremely difficult. The first concrete example of a normal number was given by an English mathematician David Gawen Champernowne. The decimal expansion of this number is obtained simply by concatenating the decimal expansions of consecutive integers: 0,1234567891011121314...99100101102... Normal numbers, as well as various ways of manipulating them so that normality is preserved, are very important due to numerous applications for example in information theory and cryptography. It turns out that if we consider just every second (or every third, and so on) digit in the expansion of a normal number, then we will always obtain an expansion of another normal number. Going further in this direction, in the early thirties, Teturo Kamae posed a question about characterization of such increasing sequences of natural numbers $n_1, n_2, n_3, \ldots$, that for *every* normal number $x = 0, a_1a_2a_3 \ldots$, the number $y = 0, a_{n_1}a_{n_2}a_{n_3} \ldots$ is also normal. (Note that if we demand this to be true only for *almost every* normal number then the question trivializes: every increasing sequence of natural numbers fulfills this weaker condition). The answer was given few years later by Kamae jointly with Benjy Weiss. They proved that the class of sequences in question coincides with the class, defined earlier by Benjy Weiss, of so-called deterministic sequences (a notion associated with entropy theory of dynamical systems).

If the sequence of digits (where the digits are positioned along a line) is replaced by another, more "spacial" configuration (for example, two-, three- or infinite-dimensional lattice), one can still reasonably define normality, determinism, and "sublattices" preserving normality. These notions may play a role in the analysis of complex phenomena with multidimensional time, for instance in control theory. We intend to study in our project (among other things) far reaching generalizations of the Kamae–Weiss theorem for miltidimensional lattices (and even more complicated networks) of digits, as well as for sequences of digits arising from other expansions of real numbers used in mathematics and other areas of science (one of such expansion is the continued fraction expansion). In cases, for which we manage to characterize the "sublattices" preserving normality, we intend to investigate their Diophantine properties (an example of a Diophantine property of a subset A of natural numbers is: if $x, y \in A$ then necessarily $x + y \in A$). In our earlier work we have proved that every normal subset of natural numbers, in addition to arbitrarily long arithmetic progressions (this fact was known before), contains also arbitrarily long geometric progressions. In other "lattices" the Diophantine properties depend on the algebraic structure of the lattice, which makes groups and semigroups the most reasonable mathematical objects to work with. This choice is reflected in the title of the project.

We are convinced that the results which we hope to obtain will attract attention of the scientific community and will contribute to the development of science, in particular in the areas linked with the theory of dynamical systems, algebra and combinatorics.