DESCRIPTION FOR THE GENERAL PUBLIC (IN ENGLISH)

Martingales form a class of stochastic processes which is of fundamental importance to the theory of stochastic integration, both in the discrete and continuous-time setting. This theory in turn is a basic tool in applications, including stochastic modelling and financial mathematics, it also plays a crucial role in other research fields such as harmonic analysis and functional analysis. From the viewpoint of these applications, it is often of interest to have tight a priori estimates for various objects intricately associated with martingales (e.g., maximal functions or square functions). Such a control is important for theoretical reasons, for example, it enables the use of limit theorems and guarantees that appropriate quantities are well-defined. The subfield of martingale theory, devoted to the study of various estimates, has been an independent and very active research area for almost 100 years now.

The purpose of the project is to study certain important classes of martingale inequalities in the wider context of noncommutative (or quantum) probability. Passing to this context in particular means that stochastic processes are no longer treated as random structures, but instead they are understood from the more general perspective of operator algebras. This branch of martingale theory has received considerable attention in the recent twenty years and many important inequalities from the classical, commutative case have been successfully transferred to the noncommutative setting. These results, in turn, have found many interesting applications in other areas, including random matrices, noncommutative harmonic analysis, operator theory and ergodic theory.

There are two important features associated with the research in the noncommutative setting. One of the major problems is the small number of technical tools which can be used in the study of various estimates: most of pointwise or trace inequalities become false when passing from the classical to the non-classical case. This often disables the direct transfer of the analysis and requires inventing clever techniques, thus making the study of various estimates much more challenging and fascinating. On the other hand, the passage from the commutative to noncommutative setting often reveals certain unexpected phenomena, e.g., certain quantities become incomparable; certain classical objects can be generalized to the noncommutative realm in many plausible ways (so there is no unique natural extension); the constants in certain estimates behave in a completely different manner after the passage to the noncommutative case, which exhibits some structural obstacles in operator theory.

The so-called Doob's maximal inequalities serve as a transparent illustration of the above issues. In the classical case, the result can be stated as follows. Let $(f_n)_{n\geq 0}$ be a martingale (a sequence of random variables possessing some structural properties) on a certain probability space. The associated maximal function, given by $f^* = \sup_{n\geq 0}|f_n|$, obviously majorizes each variable f_n, $n = 0, 1, 2, \ldots$. It turns out that the following reverse bound holds: for $1 < p < \infty$ there exists a constant c_p for which

$$(\star) \qquad \left(\mathbb{E}\sup_{n\geq 0}|f_n|^p\right)^{1/p} \leq c_p\left(\sup_{n\geq 0}\mathbb{E}|f_n|^p\right)^{1/p},$$

i.e., the collection $(f_n)_{n\geq 0}$ controls (in p-th norm) the size of the maximal function. The passage to the noncommutative setting immediately reveals a crucial problem. Namely, for a martingale $(f_n)_{n\geq 0}$ (which now becomes an appropriate collection of operators on some semifinite von Neumann algebra equipped with a trace τ) it is not clear how to define the associated maximal function: there is no default procedure leading to the supremum of operators. To overcome this difficulty, one introduces directly the "supremum norm of f", i.e., instead of defining $\sup_{n\geq 0}|f_n|$, one gives an appropriate noncommutative meaning to the left-hand side of $(\star)$. We shall not present the quite involved extension here and content ourselves with the following nice formulation in the particular case when $(f_n)_{n\geq 0}$ is positive (i.e., consists of positive operators). Namely, given $1 < p < \infty$ and a positive noncommutative martingale $(f_n)_{n\geq 0}$, there exists a positive operator a such that $f_n \leq a$ for each n and

$$(\star\star) \qquad \left(\tau(a^p)\right)^{1/p} \leq C_p\left(\sup_{n\geq 0}\tau(f_n^p)\right)^{1/p},$$

for some constant C_p depending only on p. It is also interesting to note that the (optimal) constants in $(\star)$ and $(\star\star)$ behave differently: while they share the same order $O(1)$ as $p\to\infty$, we have $c_p = O((p-1)^{-1})$ and $C_p = O((p-1)^{-2})$ as $p \to 1$. It should be also emphasized here that while the proof of $(\star)$ rests on some elementary estimates only, the argumentation leading to its noncommutative counterpart $(\star\star)$ exploits deep interpolation and duality arguments.

The project assumes the study of several classes of martingale inequalities which are important from the viewpoint of the further development of the area. These include the weighted versions of the above maximal inequalities, bounds for continuous-time martingales, the noncommutative Bellman function method and the so-called atomic decompositions.