Varieties and homogeneous ideals with symmetries

research project description for the general public

Symmetries are present in various guises in the real world. We encounter them on daily basis. Symmetrical structures and patterns are considered more aesthetic and attractive.

Symmetries can be best captured with mathematical means and their presence in mathematics is overwhelming. Usually symmetrical mathematical object enjoy much better properties than objects lacking symmetry and they are easier to handle because one has more tools at the disposal.

For example, from the two polynomials displayed below

$$9x^2 - 18xy + 9y^2 \quad \text{and} \quad 3x^2 - 17xy + 8y^3$$

the left one is obviously symmetric and it can be written as

$$9(x-y)^2.$$

From this presentation we learn immediately that its value is zero exactly for $x = y$. It is much more complicated to determine the set of zeroes of the polynomial on the right.

The aim of the project is to explore symmetries of point configurations in projective spaces and of polynomials, or more generally families of polynomials (called ideals) which have emerged recently in two, apparently not related, areas of algebraic geometry and commutative algebra.

One of these areas studies ideals of polynomials and objects derived from them, called ordinary and symbolic powers. It has been observed that there are certain regular containment relations between these powers. More precisely, if I is a homogeneous ideal of polynomials in $N+1$ variables then the containment

$$I^{(m)} \subset I^r$$

holds always whenever $m \geqslant Nr$. There is not a single example showing that this inequality is indeed strict for all values of r. In the project we will consider properties of ideals where the condition is close to be strict.

The other area is very new. It deals with objects called unexpected hypersurfaces. For example, it is well known, that there is exactly one line passing through two distinct points in the plane. For 3 general points one does not expect any line passing through them. In 2016 Cook II, Harbourne, Migliore and Nagel discovered that there are configurations of points such that one does not expect a hypersurface of certain degree passing through them with prescribed multiplicities, yet such hypersurfaces (typically just one) exist! The second objective of the project is to explain this phenomena.

The most intriguing part of the project is to explore the question why the same or closely related configurations of points appear in both problems, which seem quite unrelated at the first glance.

Finally, the advantage to conduct the proposed research in an international team lies in the possibility to combine experiences of all groups involved, look at the problems from various perspective and to combine a wide range of methods and tools. Outreaching immediate mathematical goals, effects of the proposal should foster international cooperation and increase visibility of Polish science in the international community.