Structural problems for dense and random hypergraphs (description for the general public)

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The ultimate objective of this grant proposal is to make further progress in the area of extremal and probabilistic properties of uniform hypergraphs. Within the proposal I consider 4 research tasks: #1. Dirac and Turán type problems for matchings and Hamilton cycles in hypergraphs. #2. Properties of randomly augmented dense hypergraphs. #3. Random hypergraph processes with restricted degree. #4. Mutual relations between random hypergraph models.

The first task can be classified under the extremal hypergraph theory, while the last two belong to the theory of random discrete structures. Task #2 lies at the crossroads of both areas. But all of them belong to the mainstream of contemporary combinatorics. Hypergraphs, a natural generalization of graphs, have a large potential for applications. A big role in the proposal is played by the notion of a Hamilton cycle, which has advanced from a 19th century puzzle to one of the most important objects of graph theory.

So far, unlike for Eulerian graphs and perfect matchings, there has been no characterization of the hamiltonicity at all. Even worse, the problem of determining the existence of a Hamilton cycle in a graph is NP-complete. Consequently, all hamiltonicity related questions for graphs, and even more so for hypergraphs, are hard. It applies, in particular, to tasks #1 and #2..

Under research task #1, devoted to Turán and Dirac type problems for hypergraphs, I would like to solve Erdos's Matching Conjecture from 1965 for $k = 4$ and determine Dirac thresholds for perfect matchings (for $k \geq 6$) and for tight Hamilton cycles (for $k \geq 4$). Under task #2, I am asking what is the smallest number m such that whenever m random edges are added to an n-vertex k-graph H with minimum vertex degree $\delta(H) \geq \alpha\binom{n-1}{k-1}$, the resulting k-graph contains a power of a Hamilton cycle with high probability. Within task #3, I plan to study random graph d-processes by means of 'balls-in-bins' schemes. This will allow me to apply the knowledge and methods of the theory of urn models and allocation schemes, which, hopefully, will bring some new, more refined results on the probability of saturation of such a process.

Finally, research task #4 focuses on relations between various models of random graphs. Random Graph Theory originated in the series of seminal papers by P. Erdős and A. Rényi in 1959-1968. Initially, two basic models have been studied: the binomial model $G(n, p)$ and the uniform model $G(n, m)$. Much less is known about the random d-regular graph $R(n, d)$ which is harder, since it lacks the independence of edges of $G(n, p)$ and the counting easiness of $G(n, m)$. My goal is to embed one of the Erdős - Rényi models into $R(n, d)$, so that all its monotonically increasing properties can be carried over to $R(n, d)$.