Relative entropy method for nonlinear systems of partial differential equations

Weak convergence methods for nonlinear differential equations are one of the most notable trends in modern analysis. We shall direct our attention to measure-valued solutions – a concept of solution which is seemingly extremely weak, but which proved to be surprisingly effective in the study of nonlinear problems. We will demonstrate the remarkable usefulness of the method of relative entropy for measure-valued solutions. One of its main implications is the property of measure-valued-strong uniqueness. The common belief that measure-valued solutions are feeble and highly non-unique objects makes this property all the more striking. It stands for the fact that, as long as a strong solution to a problem exists, each measure-valued solution emanating from the same initial data coincides with the strong solution.

In the light of the standstill of the notion of weak solutions for many systems of fluid mechanics, it becomes even more important to study measure-valued solutions. Indeed, for a very long time the issue of existence of weak solutions to the incompressible Euler system was out of reach. Only a recourse to the methods of differential geometry allowed for a breakthrough. In the past decade, the works of Camillo De Lellis and Laszlo Székelyhidi established non-uniqueness of solutions, and even identified ample initial data which generates an infinite number of solutions. Much hope was then pinned on admissibility criteria, like an energy inequality, which were believed to be able to select the *physically relevant* solution. However, the same authors rendered these hopes vain. They had shown that it is possible to construct infinitely many solutions satisfying such criteria. Based on the same method, called convex integration, the first proof of existence of global-in-time weak solutions to the Euler system was possible.