Stability of blow-up for the Fujita equation

Partial differential equations are an essential tool for the mathematical description of the surrounding physical reality. They provide a language in which many of the fundamental phenomena are expressed. In the description of the largest of scales we find equations of Einstein's general relativity, while in the microscale we use equations of quantum mechanics e.g. the Schr̈odinger equation. In between, partial differential equations describe processes like the heat transfer, wave propagation or the fluid flow.

Probably the most famous partial differential equation is the Navier-Stokes equation that describes the flow of an incompressible viscous fluid. It is one of the simplest examples in the family of second order semilinear parabolic equations. It is an important class of models that includes e.g. reaction-diffusion equations that have broad applications in physics, biology and economics. In contrast to linear parabolic models a singularity may form in finite time, i.e., a situation may arise whereby the solution reaches infinite values after a finite time of evolution. Clearly, every *sensible* fluid flow model should not predict infinite speeds! In spite of decades of intensive research and a handsome financial reward of 1000 000 $ (offered by the Clay Mathematics Institute) the question of whether the Navier-Stokes equation is *sensible* in the above sense remains wide open.

On the technical level, the main obstacle in solving the problem is the so-called *supercriticality*. All quantities that we know how to control (like energy for example) are supercritical with respect to the natural scaling of the equation. In practical terms this means that they are useless when it comes to the description of the flow in the microscale. However, so far we know of nothing that would prevent a singularity from hiding exactly there. The phenomenon of supercriticality is by no means restricted to the Navier-Stokes equation. It often occurs whenever we model a highly unstable, turbulent process. It gets worse still - we cannot tame supercriticality even in equations that are formally much simpler like the Fujita equation put forward in the sixties as a simplified model for "things that may go wrong in Navier-Stokes". Once we learn how to handle supercriticality in the Fujita equation we will gain an important insight into subtleties of analysis of other supercritical models.

This project is intended to deliver new mathematical techniques enabling us to exclude the possibility of occurrence of so-called type II blow-ups for the Fujita equation. These are subtle and highly unstable phenomena supercritical with respect to scaling that may lurk in the microscale of the Navier-Stokes regularity problem. Analysis of such singularities requires sharp and dedicated tools. The Fujita equation provides a perfect environment for confirming or refuting conjectures and sharpening techniques designed to tackle singularities of this type in semilinear problems. In the eighties Giga and Kohn proved that in some cases the type II singularity may be excluded through a method that used the scaling symmetry of the equation to define a special frame of reference in which the potential singularity assumes a profile that cannot possibly exist. The efficiency of this method ends however the moment we deal with cases where profiles of the required shape are known to exist. The key idea in this project is to replace (whenever necessary) the non-existence argument with the instability argument, i.e., showing that in spite of the profile's existence the singularity of this shape cannot occur because its instability prevents the solution for approaching it.

The proposed method comprises a universal and a specific aspect. The universal one may be transferred to theories of other related equations. The specific side addresses the concrete form of the equation at hand. The universal aspect extracts information form structural considerations (like symmetries of the equation) in order to define mathematical objects that describe potential singularities. The specific aspect provides a concrete form of such objects specific to the given problem.