# Toeplitz and Hankel operators between distinct Hardy spaces

The aim of the project is the comprehensive study on the Toeplitz and Hankel operators acting between different spaces of analytic functions. To clarify the idea of the proposal we recall two basic definitions of Toeplitz and Hankel matrices. Let $(a_n)_{n=-\infty}^{\infty}$ be a sequence of complex numbers. The Toeplitz matrix has a form

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

while the following matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ a_3 & a_4 & a_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is called the Hankel matrix.

The above matrices are closely connected to specific operators on spaces of analytic functions. Let $H^2(\mathbb{T})$ be a Hilbert–Hardy space on the unit circle. It is a well known fact that any bounded operator on $H^2(\mathbb{T})$ such that it's matrix representation (relative to the standard basis in $H^2(\mathbb{T})$) has a form of Toeplitz or Hankel matrix, can be respectively written as

$$T_a\colon f \mapsto PM_a f, \quad H_a\colon f \mapsto PM_a Jf.$$

In the above formula $M_a$ is the multiplication operator by a function $a \in L^\infty$ ($a$ is called the symbol of $T_a$ or $H_a$ or $M_a$), $P$ is the classical Riesz projection (i.e., $P = (I+H)/2$, where $H$ is the Hilbert transform), and the operator $J$ is given by $Jf(t) = t^{-1} f(t^{-1})$ for $t \in \mathbb{T}$. Moreover, the sequence $(a_n)$ is the sequence of Fourier coefficients of the symbol $a$. Operator $T_a$ is called the Toeplitz operator and $H_a$ – Hankel operator.

The above operators play an important role in the operator theory and harmonic analysis. The systematic study of them started in the first half of XXth century, while, of course, a special cases (for example the Hankel matrix) were investigated much earlier. Nowadays properties of these operators are very well investigated, but their genearlizations and connections with another fields of mathematics are still a current research object.

However, most of the studies (in fact almost all) refers to the situation where $T_a$ or $H_a$ act from a specific space of analytic functions to the same one (i.e. the so-called algebraic case). **The novelty of the project is that the considered operators act from one space of analytic functions to a different one.** In particular, the unbounded symbols $a$ of operators $T_a$ and $H_a$ are allowed.

The principial goal of the project is to answer basic questions originating from the theory of Toeplitz and Hankel operators on $H^2(\mathbb{T})$, as well as to consider genuine problems which will appear when studing such general and untouched matters as investigation of the mentioned operators between distinct spaces. It should be pointed out, the reasearch team will not be focused on seeking expected generalizations of the known theory, but rather will be looking for new phenomena, which should appear in the proposed situation. When possible, we will work with general Hardy spaces, but the anticipated results will be new even for the classical case of $H^p(\mathbb{T})$ spaces.

We will use methods of functional, harmonic and complex analysis, but the theory of interpolation and function spaces will play a particular role as well. Our general approach will require to discover new methods, since most of the known tools from the algebraic case will not be available. We believe also that our point of view and developed methods will inspirate analogous research for other types of operators, which, so far, are considered in the algebraic case only.