COMBINATORICS OF SYMMETRIC POLYNOMIALS

Modern mathematics is getting more and more specilized and many current research programs are lying entirely in a tiny fragment of mathematics accessible only for highly specialized experts. However, from time to time mathematicians discover some ideas and objects that connect many of these seemingly unrelated fragments of mathematics. **Symmetric polynomials** are one of the examples of such objects. Their definition is simple - they are polynomials such that if any of the variables are interchanged, one obtains the same polynomial. For example:

$$f(x_1,x_2,x_3) := x_1 \cdot x_2 \cdot x_3 = f(x_1,x_3,x_2) = f(x_2,x_1,x_3)$$
$$= f(x_2,x_3,x_1) = f(x_3,x_1,x_2) = f(x_3,x_2,x_1)$$

is a symmetric polynomial, but

$$f(x_1,x_2,x_3) := 2x_1 + x_2 + x_3 \neq 2x_2 + x_1 + x_3 = f(x_2,x_1,x_3)$$

are not. It turns out that some symmetric polynomials are surprisingly remarkable– **they appear naturally in many different fields of mathematics and physics**, and their beautiful and rich structure lead to many important discoveries in those fields. One can ask – *why their algebraic structures are so special? What makes them to be the objects appearing so widely?*

One of the possible answers for the above questions is the following: all these polynomials have one thing in common - their structure can be described in terms of some concrete objects whose enumeration and properties are accesible by studying patterns explaining the nature of these objects. *What kind of objects we are talking about?* Let us look on some examples: Figure 1(a) shows the tiling of a given staircase shape by tiles of a ribbon shape of size 5. Here is another example: Figure 1(b) shows the filling of a staircase shape by distinct numbers $1, 2, \ldots, 24$ in a way that in the first, second, fourth and fifth rows there is at least one *inversion*, which is a pair of entries such that the left entry is bigger than the right one. Studying patterns in these objects we can answer the following question: *in how many ways we can tile a given staircase shape by tiles of a ribbon shape of size* 5*? In how many ways we can fill a given staircase shape by distinct consecutive numbers if we require that in the first, second, fourth and fifth rows there is at least one inversion?* The branch of mathematics which study such finite configurations and rules describing their structures is called **combinatorics**. **Algebraic combinatorics** is a field which uses combinatorial objects and methods to describe some abstract, algebraic structures such as algebraic properties of some special symmetric polynomials.

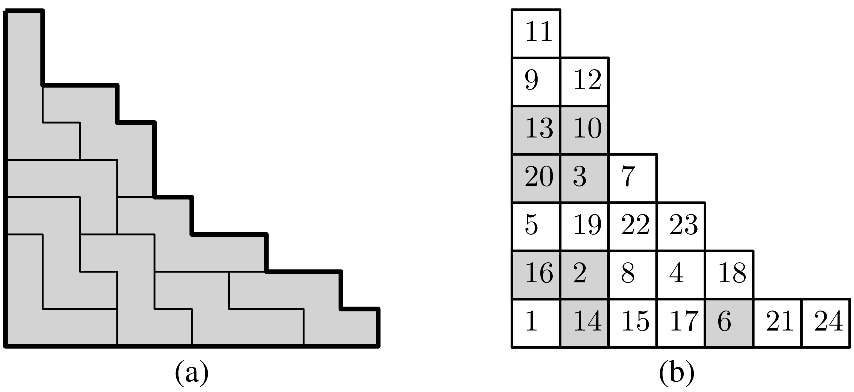

Figure 1. Figure 1(a) presents a tiling of a staicase shape by ribbons of size 5. Figure 1(b) shows a filling of a staicase shape by distinct numbers $1, 2, \ldots, 24$, and some inversions are highlighted in grey.

The main goal of this project is **to discover and study combinatorial structures** of the similar nature as the ones described above and **to explain beautiful and mysterious properties of some special symmetric polynomials**. As a result we plan to use these concrete combinatorial structures **to obtain some new results in seemingly unrelated fields of mathematics**.