Radicals of crossed products of universal enveloping algebras and dimensions of subalgebras of matrix algebras: description for the general public

All rings and algebras, unless stated otherwise, are assumed to be associative.

Matrix algebras with entries coming from a division ring or a field are one of the most important object in mathematics. The validity and the multiplicity of places where matrices appears in the modern science is well known. We can mention here at least the following: analysis and geometry, probability theory and statistics, symmetries and transformations in physics, linear combinations of quantum states, medicine, economy.

It is well known that any finite dimensional associative algebra A is a direct product of matrix algebras up to some "bad" part called the radical of A (maximal nilpotent ideal of A). More precisely, if we denote the radical of A by $\mathrm{rad}(A)$, then factor algebra $A/\mathrm{rad}(A)$ is isomorphic to a direct product of matrices over division rings. This beautiful fact is a motivation to study "bad" parts of associative rings and infinite dimensional associative algebras. It occurs that if we do not assume that we work with a finite dimensional associative algebra, related pathologies which can appear are of manifold nature. It leads to definitions of many kinds of radicals in this general case.

In the project we are going to investigate some radicals of crossed products of universal enveloping algebras. More precisely, we will consider a Lie algebra L, its universal enveloping algebra $U(L)$ (it is associative algebra) and a ring R. Then we study associative algebra - denoted by $R * U(L)$ and called crossed product of $U(L)$ - whose definition is based on an action of $U(L)$ on R as derivations. Special examples of investigated structures are Weyl algebras and differential polynomial rings.

The project is divided into two parts and now we want to discuss second of them.

The main objects here are subalgebras of the algebra $M_n(K)$, over a field K. For mentioned class of algebras we will study their subalgebras A and their dimensions as linear spaces. Presented research will be related to subalgebras satisfying some polynomial identities and by this is a part of PI theory of associative algebras. Because this is a summery for the general public we can say that we will consider subalgebras A of $M_n(K)$ which satisfy a fixed polynomial identity

$$f(x_1,\ldots,x_m)=0, \text{ where } f(x_1,\ldots,x_m) \text{ is a polynomial over } K \text{ in non-commuting variables}$$

which means that for all $a_1,\ldots,a_m \in A$, $f(a_1,\ldots,a_m)=0$. As an example we can mention the identity $xy-yx=0$ which leads us to questions on commutativity of subalgebras. If we take $M_2(\mathbb{Q})$, then this is easy to see that the subalgebra

$$A=\left\{\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a,b\in\mathbb{Q}\right\}$$

of $M_2(\mathbb{Q})$ satisfies $xy-yx=0$.

Notice, that the dimension of A over $\mathbb{Q}$ is equal to 2. By well known results (Schur, Jacobson, for any n and $M_n(K)$), for subalgebras of $M_2(K)$, satisfying $xy-yx=0$, the dimension can not be larger than 2.

Since the dimension of $M_n(K)$ over K is equal to n^2, any subalgebra of $M_n(K)$ has finite dimension. Put it in simple, our main goal in this part of the project is to find some functions which values depend somehow on n and fixed polynomial identity $f=0$, and these values give us information on the maximal possible dimension of subalgebras satisfying mentioned identity.