Geometry of some K3 surfaces with many rational curves

description for general public

Objectives of the project

This project is devoted to algebraic geometry. We want to determine maximal number of lines on sextic (resp. octic) $K3$-surfaces. Furthermore, we intend to find explicit equations of all such surfaces that carry many lines. We do not restrict our attention to complex smooth surfaces but want to study the ones with singularities and the ones defined over a field of positive characteristic as well.
Another objective of the project is construction of quartic $K3$-surfaces with many smooth rational degree-d curves, where $d \geqslant 2$. The idea of the construction is to consider the Kummer surface of an abelian surface (i.e. we consider a two-dimensional torus, glue each pair of its antipodic points to one point, replace each of the sixteen points where the new surface is not smooth by a line and arrive at the Kummer surface) and study bundles on the Kummer surface. If the original abelian surface has many symmetries, the resulting Kummer surface contains many curves, some of which may become smooth rational degree-d curves. In order to recognize the right curves, one has to apply techniques of modern algebraic geometry.

Reasons for choosing the topic

One can show that no general formulae for roots of polynomial equations of high degrees can be found. Over centuries mathematicians tried to circumvent this problem by studying the properties of sets of solutions of systems of algebraic equations. In particular, algebraic geometers managed to divide such sets of solutions into various classes of solution sets with similar properties. One of such classes are $K3$-surfaces.

$K3$-surfaces can be seen as two-dimensional analogues of elliptic curves. They are also of interest for theoretical phycisists (as Calabi-Yau varieties of small dimension). Among smooth $K3$-surfaces only three types can be obtained as complete intersections in projective space. These are quartics, sextics and octics. The geometry of quartics with many lines is now pretty well understood. The major progress took place over the last five years. Although part of the results on quartics were obtained by computer-aided symbolic computations, most properties were derived by human mind using algebraic geometry and algebra. The quartics had one special property that simplified the arguments: they were given by one equation. In the project we want to arrive at analogous results for two remaining natural classes of $K3$-surfaces. Sextics and octics are no longer hypersurfaces, which is an additional difficulty. Moreover we want to study the singular models as well.

Our interest in constructing explicit examples and finding explicit equations is motivated by the fact that surfaces with many lines can be used to construct various mathematical objects with unexpected properties. Therefore such explicit, well-understood surfaces with special properties are very valuable for testing conjectures and seeking unexpected relations between various mathematical objects.

Research to be carried out

We will apply various modern techniques of algebraic geometry. One of the ideas of the project is replacing the $K3$-surface in question by a $K3$-surface given by one equation (a hypersurface) in such a way that we can read various properties of the original surface from the geometry of the hypersurface.
Another idea is change of base field - i.e. replacing the set of considered numbers by a smaller one without changing abstract properties of the set of solutions. In some cases we will also use symbolic computer-aided computations.

Finally we want to analyze the discrepancy between the maximal value of the number of lines that we intend to compute and the bound given by the orbibundle Miyaoka-Yau-Sakai inequality. This is motivated by our interst in general results on numbers of lines/smooth rational curves on projective surfaces.