Project Description for the general public:

Open problems in valuation theory in positive characteristic

The research project is connected with two deep, longstanding open problems in mathematics. They come from two quite different areas: algebraic geometry and mathematical logic.

Algebraic geometry studies systems of polynomial equations. They appear in all sorts of problems in science and technology that have an impact on our daily life. For instance, weather forecasts need huge systems of differential equations, which in turn are based on polynomial equations. The same is true for crash-test simulations which car producers run in order to improve the safety of their cars; it is too costly to always smash a new car into a wall when its safety has to be tested.

A main obstacle in handling such systems is the existence of “singularities”. These are points where the solutions of the system of equations do not depend in a well-behaved way on its parameters. But chaotic behaviour is not what we want to see when we try to predict the weather.

Many excellent mathematicians have worked to achieve “resolution of singularities”, that is, the transformation of a given system into a new system that has no singularities. In 1964, Heisuke Hironaka obtained resolution of singularities in an important particular case (“characteristic zero”). For this work he received a Fields Medal (the equivalent of the Nobel Prize for mathematicians). In the Japanese society, which is interested in and honours science, Hironaka has been since then a well known celebrity.

Many attempts have been made to settle the remaining case of “positive characteristic”, but the problem is still open. If one cannot solve a problem globally (resolve all singularities) one tries to solve it locally. This is called ”local uniformization” and may be understood as handling one singularity at a time. But one has to trace what becomes of this singularity under the transformation. Oscar Zariski, who was the teacher of Hironaka, proposed in the 1930’s to do this using “valuations”. He proved in 1940 that local uniformization can be done in characteristic 0. But the problem is still open whether it can be done in positive characteristic.

In mathematical logic, one prominent question is whether certain mathematical structures are “decidable”. This means that there is an algorithm which (in principle) can decide whether a statement formulated in the mathematical language in which we talk about the structure is true or not. For instance, the real numbers and the complex numbers are decidable, but certain structures arising from number theory (i.e., the theory of the integers) are not.

Number theory studies prime numbers, and connected with a prime number p is a “p-adic valuation” and a structure called the “field of p-adic numbers”. In 1965 James Ax, Simon Kochen and Yuri Ershov showed that it is decidable. Their work showed that another structure, also equipped with a valuation, is closely related. But while the former is in characteristic 0, the latter is in positive characteristic, and the problem whether it is decidable has remained open till the present day.

Valuation Theory is the area of expertize of the principal investigator of the proposal. The two big open problems of local uniformization and decidability at first do not seem to be related with each other, but in his work he has shown that they have common roots in valuation theory. The obstacle to the solution of both is a nasty phenomenon which is aptly called the “defect”. Through a detailed study of the defect the principal investigator wishes to contribute to the solution of the two open problems.

The project will have a strong component of building an international research network of cooperation between young mathematicians and experts.