Description for the general public

The fact that the methods of complex analysis may be used for proving theorems of arithmetics is far from obvious. The first attempts in this direction may be found in the works of Euler, and later Dirichlet. Historically the first treatise that made systematic use of analytic methods in the study of the distribution of prime numbers was the famous memoir of B. Riemann of 1856. There the author considered the function $\zeta(s)$, now called the Riemann zeta function. He considered it in full generality, i.e. as a function of the complex variable s. He described its basic properties and set the direction of future research. In particular he discovered the significance of so called non-trivial zeros of the zeta function and he formulated the famous "Riemann Hypothesis". Subsequent generations of mathematicians were able to discover a number of special functions similar to $\zeta(s)$, called L functions. They have a number of characteristic properties, e.g., in the half-plane $\Re(s) > 1$ they can be defined as sums of Dirichlet series, they have a meromorphic continuation to the whole complex plane and they satisfy a Riemann-type functional equation with multiple gamma factors. An axiomatic approach to the theory of L functions is due to A. Selberg. In 1989 he defined a class of Dirichlet series, now called the Selberg class. To describe a structure of this class is a problem of fundamental importance for number theory. There exist several open and quite brave conjectures in this regard. Two of the most important ones are the Degree Conjecture and the conjecture that the so called "general converse theorem" holds. Taken together they imply that the Selberg class equals—up to a certain normalization—the class of Langlands functions related to automorphic representations of the groups GL_n. The aim of the proposed project is to

(A) achieve progress in the description of the Selberg class and the extended Selberg class, mainly in the case of degree 2,

(B) to extend the domain of application of Selberg class functions in number theory, in particular to achieve progress in the description of factorization properties in analytic monoids, and to apply analytic methods to the construction of computational number theory algorithms and to the complexity analysis thereof.

The main proof method in (A) above will be the theory of non-linear twists of L-functions, developed in earlier papers of J. Kaczorowski and A. Perelli. It has already proved to be very effective in studying the structure of the Selberg class. It seems that the method of twists, particularly of the multidimensional ones, has a great potential. Its significance and range of applications have not yet been completely explored.

Methods to be used in (B) belong to the modern toolset of analytic number theory (asymptotic expansions, so-called explicit formulae, complex integration, etc.). The key to the understanding of the deeper problems of quantitative factorization theory will be a sufficiently precise description of the independence of distinct L-functions, in particular showing the existence of singularities of combinations of logarithms and complex powers of L-functions from the Selberg class.