Arithmetic and geometry of parameter spaces of line arrangements

description for the general public

Line arrangements are very popular objects of research in combinatorics and geometry. By a line arrangement we understand a finite set of lines on the plane with all singular points i.e. points where two or more lines meet. This definition is easy to explain and the problems concerning configurations of lines are easy to formulate in an elementary way. Usually difficulties come up when we look for solutions. This makes the subject interesting, since mathematicians like challenges. Configurations of lines and their intersection points have been studied quite extensively with respect to the many distinct issues in mathematic, in the past and today.

The most famous problem concerning the arrangements of lines is the orchard problem formulated in 1821 by Jackson in the following way

Fain would I plant a grove in rows,
But how must I its form compose
With three trees in each row;
To have as many rows as trees;
Now tell me, artist if you please;
This is all I want to know.

This puzzle is strongly connected with the well known Sylvester–Gallai theorem, which says that for the finite set of non-collinear points in the Euclidean plane there exists a line including exactly two of these points (a so called *ordinary line*). First let us notice, that the Sylvester–Gallai theorem is false over the complex numbers. As an example we can give the famous Hesse configuration with 12 lines arranged such that there exist exactly 9 quadruple intersection points and every line contains exactly three of them. Thus we have a line arrangement with no ordinary lines over complex.

And here question arises, what is the minimal number of such lines with exactly two points for real configurations of points? Along these lines it is also natural to wonder what is the maximal number of triple intersection points, which we could achieve in configuration of lines? Studying lines arrangements with respect to the number of ordinary lines and number of triple points is a popular subject of research in recent years. Configurations with relatively large number of triple points turned out to provide counterexamples to some containments between ordinary and symbolic powers of ideals in algebraic geometry. The first published counterexample, over complex numbers, was given by set of points coming from the dual Hesse configuration. Counterexamples over real and rational numbers also comes from some line arrangements. More precisely from the Böröczky configuration of 12 lines with 19 triple points, published by Füredi and Palasti in 1984. The only known rational counterexample comes from a modified Böröczky construction. Interestingly this construction of 12 lines depends on only one parameter, thus we have a whole family of arrangements over the rationals, in contrast to Böröczky construction for 15 lines, which can be never realized over rational numbers.

This result is the main motivation for my studies in this area. I am interested to find out, what is the behaviour of remaining Böröczky configurations and some other arrangements of lines. Mainly I am interested to establish what are their parameter spaces and which of these constructions can be carried out over rational numbers. The parameter space can be taken as a divisor in $(\mathbb{P}^1)^k$ or as a singular plane curve. Both of this paths of considerations leads to interesting conclusions, connecting the behaviour of some algebraic objects with properties of all configuration. I would be also interested to find more rational counterexamples for mentioned containment. It is surprising, that up know we know only this one coming from 12 lines of Böröczky, although the search in this subject are very intensive for the last years.

This path of research was recently taken among other by me for configurations of 12 and 15 lines of Böröczky and I would like to extend it by much more configurations.