Heat kernels: construction and estimates

The heat equation $\partial_t = \Delta$ plays a fundamental role in physics, since it can be used to model the distribution of heat. The fundamental solution of this equation is the function $g(t,x) = (4\pi t)^{-d} e^{-|x|^2/4t}$, called the Gauss-Weierstrass kernel. It means that given the distribution of temperature at the initial moment $t = 0$, using the function $g(t,x)$ we can determine the temperature at any time $t > 0$ and any space point x.

The function $g(t,x)$ is also very important in the theory of stochastic processes, which provide mathematical models of random phenomena. The first prominent example of such processes is the Wiener process, known also as Brownian motion. It was proposed in 1900 by L. Bachelier to examine prices on the stock market, used later by A. Einstein and M. Smoluchowski to describe the Brownian motion (chaotic movement of a particle in fluid, e.g., a pollen in water) Under some assumptions on the model, the function $g(t,x)$ characterizes the Wiener process. In particular, it allows to obtain probability of finding a particle in a chosen region of the fluid after given time t. For that reason we call $g(t,x)$ the transition density of the process.

Another important example of a stochastic process is the Poisson process. A value of that process at time t can be interpreted as a (random) number of signals, or more general events, that occur up to moment t. It was used already in XX century to study the operation of telephone exchanges and analyze the capital of insurance companies. It is noteworthy that the realization (or the so-called trajectories) of the Poisson process have jumps. They occur whenever the signal comes. On the other hand, trajectories of the Wiener process are continuous.

Nevertheless, both processes are representatives of the same larger class of stochastic processes known as Lévy processes. Trajectories of most of them have jumps, which are described by the Lévy measure (jump measure), and their structure is much more complicated than that of the Poisson process. We want to emphasize that if a Lévy process has a transition density $p(t,x)$, it might not be easy to find its explicit formula. In such case precise estimates, asymptotics and regularity properties of $p(t,x)$ are of importance. The connection, indicated above, between the heat equation and transition density of the Wiener process is not accidental. Similar relation holds true between equations of the form $\partial_t = A$, where A is the generator of the Lévy process, and the distribution of the Lévy process. In this context we could discuss even larger class of processes, named Markov processes. Generators of (Markov) processes, whose trajectories have jumps will be called non-local operators, and we say heat kernels for their transition densities.

There has been an increasing interest in non-local operators since they allow to create more precise models in physics, chemistry, biology and economics. They are used to describe evolution of nonlinear waves, fluid dynamics or in the study of polymers. Some of them are created on spaces other than Euclidean, to allows of the study of structures like crystals.

Project objectives

We will study heat kernels of Markov processes with discontinuous paths. Equivalently, we discuss a class of pseudo-differential operators with non-zero non-local term. We will (1) construct and estimate the corresponding operator semigroups on the whole Euclidean space, (2) obtain estimates and asymptotics for semigroups on domains, and (3) find asymptotics behaviour and estimates for heat kernels on discrete structures. In particular we will study heat kernels of non-local operators with non-symmetric Lévy measures, Dunkl operators and subordinators. We will consider processes with non absolutely continuous Lévy measures in domains and we will obtain asymptotic behaviour of the Dirichlet heat kernels near the boundary of considered domains and find asymptotics for the spectral heat content. We will also study random walks on discrete structures, for instance on affine buildings and topological crystals.