Description for the general public

A fundamental principle in mathematics and physics is the exploitation of symmetry. In geometry, it was in particular Felix Klein's Erlangen programme from 1872 that stressed its importance. A central notion became that of a symmetry group; it is formed by all transformations of a space that leave all its geometric properties unchanged. When Willhelm Killing discovered five previously unknown simple mathematical objects that could be symmetry groups, it was a natural problem for mathematician at the time to find geometric structures with those symmetry groups. For the smallest of these, the so-called exceptional group G_2, this was accomplished, independently, by Élie Cartan and Friedrich Engel in 1893: they found two different geometric structures on five-dimensional spaces with G_2 as their symmetry group.

This research project aims to investigate generalizations of the two geometric structures found by Cartan and Engel. Although most of these generalized structures have no symmetries at all, their geometry is still closely linked to the exceptional group G_2. To give an idea what these geometric structures are, we describe a class of them on a simple mechanical model:

Consider a ball rolling on another ball without slipping or twisting. A position of this mechanical system can be specified by five coordinates: two for the point of contact on the first ball, two for the point of contact on the second ball, and one for an angle that measures the mutual orientation of the balls at the point of contact. So this is a 5-dimensional space. A rolling motion can be thought of as a curve in this 5-dimensional position space. The curve cannot be arbitrary, since we restrict the possible movements: we do not allow the ball to slip or twist. Mathematically, these constraints describe a geometric object called a rank 2 distribution. If we further assume that the radii of the balls are distinct, then this distribution has a particular property, which in practice tells us that given any two positions of the two ball system one can get from one position to the other by rolling without slipping or twisting. (This is not the case if the radii are equal!) Distributions of this type are examples of the geometric structures this project aims to investigate. We can also describe the distribution found by Cartan and Engel on this mechanical model. It is distinguished by the non-obvious fact that it is more symmetrical than the others: For all ratios of radii the rolling distribution has a 6-dimensional symmetry group, except if the ratio is 1:3, then the dimension jumps to 14 and the local symmetry group is precisely the exceptional group G_2.

The geometric structures studied in this research project can be abstracted also from more complex mechanical systems that arise naturally, for example, in the field of robotics; we hope that understanding their properties will lead to interesting applications in this area. Most importantly, the geometric structures appear in a number of research questions in pure mathematics and mathematical physics that have been studied intensively in recent years and have lead to fascinating developments. The main concrete objective of the research project is to answer specific questions about G_2-related geometries. In doing so, we expect to develop new mathematical tools and thus contribute to a wider understanding of modern non-Riemannian geometry.