"Combinatorial methods" "in algebraic geometry and commutative algebra"

Mateusz Michałek, Instytut Matematyczny PAN

The aim of the project is to solve or make progress in many interesting problems and conjectures on the edge of algebra, combinatorics and geometry. The intersection of the mentioned fields has become recently a particularly fast developing area. It allows applications of various methods. The interplay between methods of algebraic geometry and combinatorics brings particularly strong results. On one hand algebraic methods often have global character, on the other hand combinatorics is often called the 'nanotechnology' of mathematics.

The project consists of four parts. In each of them we point out specific aims.

0.1. **Hyperbolic and completely monotone polynomials.** A a homogeneous polynomial P with real coordinates is called hyperbolic with respect to vector v if for any vector y there is $P(v+iy) \neq 0$. A polynomial is called hyperbolic if it is hyperbolic with respect to some vector. A function f on $(0,\infty)^n$ is completely monottonic if for every vectors $u_1, \ldots, u_k \in (0,\infty)^n$ there is:

$$(-1)^k \frac{\partial^k f}{\partial_{u_1} \cdots \partial_{u_k}} \geqslant 0.$$

We intend to study combinatorial and algebraic properties of hyperbolic polynomial P. In particular, conjectures about complete mononicity of functions $P^{-\beta}$.

0.2. **Inequalities for volume of the difference set.** The difference set $D(K)$ of a convex set $K \subset \mathbb{R}^n$ is defined as the Minkowski sum

$$D(K) = K - K = \{x - y \mid x, y \in K\} \subset \mathbb{R}^n.$$

Bounding the volume of the difference set is an important problem in convex geometry, that has applications in geometric number theory. The fundamental inequalities are:

$$2^n \operatorname{Vol}_n(K) \leqslant \operatorname{Vol}_n(D(K)) \leqslant \binom{2n}{n} \operatorname{Vol}_n(K).$$

Our aim is to prove a conjecture of Godbersen, which is a strengthening of the right side of the above inequality.

0.3. **Chromatic polynomial of a matroid.** If we substitute -1 to the chromatic polynomial of a graph, we get the number of its acyclic edge orientations. This is an example of a reciprocity theorem. Our aim is to find and prove a reciprocity theorem for the chromatic polynomial of a matroid (i.e. a polynomial whose value at n is the number of proper (elements of the same color form an independent set) colorings of a matroid with n colors).

0.4. **An extremal problem on crossing vectors.** The aim of the last part is to make progress in a problem from the area of extremal combinatorics. The question is, how many vectors there can be in $\mathbb{Z}^w$, which are pairwise 1-crossing, but not k-crossing. We believe that the problem has an algebro-geometric side. Moreover, it has important implications for the theory of partial orders.