Lie systems and selected Lie theory methods in differential equations

Partial and ordinary differential equations account for the most powerful tools in describing properties and evolutions of physical systems. Moreover, they represent a substantial part of the contemporary mathematical sciences. Most of the literature in differential equations is devoted to the analytical description of their properties and solutions, leaving in big part aside their geometric study. Notwithstanding, relevant differential equations are generally related in this or other way to a geometric structure (cf. Lie symmetries) which is veiled in its analytical form. For instance, one of the most important nonlinear differential equations, the Riccati equation, can be understood as a linear combination (with time-dependent coefficients) of three vector fields on the real line, spanning a Lie algebra of vector fields.

Lie theory, named after the Norwegian mathematician Sophus Lie who pioneered it, was initially interpreted as a Galois theory for differential equations and only posteriorly evolved into a theory of algebraic-geometric-differential objects nowadays known as Lie groups and Lie algebras. These concepts became one of the most fundamental notions and tools in mathematics and physics as a whole. Lie groups, Lie algebras and their more general counterparts, the Lie algebroids and Lie grupoids, appear in the study of symmetry of differential equations, reductions, and problems of integrability. There are, however, sometimes hidden but equally important connections of both theories.

One of the most interesting properties of this kind, for systems of first-order differential equations, is the existence of a, generally nonlinear, *superposition rule*. This superposition rule is generally understood as a function producing new solutions out of the knowledge of a particular set of known ones and some parameters. In general, a superposition rule allows one to retrieve the general solution from the knowledge of a finite family of generic 'independent' particular solutions. This is a good substitute of integrability in cases in which there is no hope for solving a differential equation explicitly. A first-order system of linear differential equations constitutes the simplest differential equation admitting a superposition rule: every solution is a linear combination of a maximal set of linearly independent particular solutions. Sophus Lie proved that a system of first-order differential equations admits a superposition rule, i.e., in the nowadays terminology it is a *Lie system* (like for instance the Riccati equation), if and only if it can be described as a time-dependent linear combination of vector fields spanning a finite-dimensional Lie algebra.

Our research project concerns as well far reaching generalizations of Lie systems (e.g. so-called quasi-Lie systems or Lie systems based upon different types of Lie algebroids) in the context of additional geometric structures (Poisson, symplectic, Dirac, etc.). Problems concerning symmetries and integrability in stochastic, quantum and supersymmetric contexts, i.e., systems of differential equations on supermanifolds, will also be part of our interests. These topics are scarcely studied and understood so far. Another brand new topic in this context will be the aim of understanding and describing geometric discretizations of Lie systems preserving the existence of superposition rules. Undoubtedly, these problems lay among the most important problems in the intersection of interests of physics and mathematics which are related to the theory of differential equations.

To accomplish this project, we succeeded in organizing a research team unifying most world-wide relevant groups working on the topic from four centers: IMPAN, University of Zaragoza, CRM- University of Montreal, and Polytechnic University of Catalonia. Each group provides a slightly different approach and competencies, which enables our research group to become complete and comprehensive. We believe that the proposed research team, its experience and competencies, ensures our success.