Arithmetic properties of formal groups SUMMARY (popularized version)

Masha Vlasenko

Formal groups have wide applications in number theory, algebraic geometry and algebraic topology, ranging from congruences for the coefficients of modular forms and local class field theory to extraordinary K-theories and homotopy groups of spheres. In this project we will develop new techniques in the theory of formal groups and apply them to important problems in other fields.

To explain one of the key ideas underlying our research proposal, let us consider a simple example. There is a natural way to turn the set of points of an elliptic curve $E : y^2 = x^3 + a_1x^2 + a_2x + a_3$ into an abelian group, where the role of identity is played by the 'invisible' point $\mathcal{O}$ at infinity. The differential $\omega = dx/y$ is invariant with respect to the group operation. Choose a local coordinate near $\mathcal{O}$, e.g. $t = x/y$, and expand the invariant differential as a power series $\omega = \sum_{n=0}^{\infty} b_n t^n \, dt$. Assuming that the coefficients a_1, a_2, a_3 are rational numbers, for all but finitely many primes p (we exclude those which occur in the denominators of a_i's and b_n's or in the discriminant of E) one has the following dichotomy. If $p | b_{p-1}$ then the curve has p points over $\mathbb{F}_p$, the finite field with p elements. If $p \not| b_{p-1}$ then the ratios $b_{p^k-1}/b_{p^{k-1}-1}$ converge in p-adic topology to a p-adic unit $u \in \mathbb{Z}_p^\times$, and the number of points of E over $\mathbb{F}_p$ differs from p by $u + p/u \in \mathbb{Z}$. We see that the expansion coefficients of ω 'know' how many points are there on the curve over each finite field.

The above example can be generalized to the Artin–Mazur formal groups associated to cohomology of algebraic varieties. More generally, we will study p-adic analytic properties of expansion coefficients of invariant differential forms and symmetric functions on formal group laws. Though this subject has been studied since 1980's, our project is based on observations that were brought up recently from the side of mirror symmetry.

Originally discovered by physicists, mirror symmetry remains one of the central research themes binding string theory with algebraic geometry and arithmetic. Numerous examples show that expression of the mirror map in so-called canonical coordinates has rich arithmetic properties, such as modularity and various congruences. This expression involves particular solutions to a Picard–Fuchs differential equation of a family of Calabi–Yau manifolds near a singular point. We plan to apply techniques of the theory of formal groups to study p-adic analytic properties of such solutions.

Our research topic has a link with the theory of p-adic L-functions. To a formal group law one associates a sequence of numbers which generalizes the Bernoulli sequence in a natural way. We should be able to apply our techniques to p-adic interpolation of those generalized Bernoulli numbers. We hope to contribute to the study of special values of p-adic L-functions. The latter is a technically challenging subject, where an ultimate goal would be to treat some cases of the Gross–Stark conjecture.