Multidimensional Selberg sieve, almost prime k-tuples and primes in arithmetic progressions

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Prime numbers have fascinated mathematicians since ancient times. Today, the concept of primarity plays the central role in analytic number theory. There are many conjectures focused around them like twin prime conjecture or the more general Hardy−Littlewood k-tuple conjecture, which implies that for example there exist infinitely many n for which every elements of the set $\{n, n+2, n+6, n+8\}$, which is a special case of an admissible tuple, are prime. This conjecture lays far beyond the reach of current methods, although we are able to weaken it a bit and then it becomes vulnerable to our attacks. For example Maynard proved that every admissible triple (like for example $\{n, n+2, n+6\}$) contains numbers which has only 7 prime factors in total for infinitely many n. There exist similar results also for different integers k.

Our first goal is to study, whether the multidimensional Selberg sieve - a method which gave numerous fantastic results in recent years - is strong enough to improve current records; we would like to find the lowest possible r_k for which every admissible k-tuple contains at most r_k prime factors in total for infinitely many n.

Results described above are usually proven via strong tools enabling us to tightly estimate the number of primes in arithmetic progressions. In the context of sieve theory, the classic example is Bombieri−Vinogradov theorem. Our second goal is to study numerically, how exaclty this theorem can by enhanced. We plan to focus on Elliott−Halberstam conjecture mostly; we would like to study, what is the asymptotic behaviour of combined error term described by this hypothesis which can be understood as a function of one variable. By this approach, we can also prepare a diagram of this function for big arguments and discover some interesting and unkown properties.

Studying Elliott−Halberstam conjecture and almost prime k-tuples is important for the same reason - it can deliever a knowledge about true primes. In the first case it is very straightforward since Elliott−Halberstam conjecture is about distribution of primes in arithmetic progressions; in the second case the point is that almost primes reminds primes very much. They can be seen as a training ground for methods which are created in order to study the structure hidden behind the primes. This way, we discover the limits of our techiques like the parity problem. Perhaps, these issues will be better understood in the future and finally solved.