Flag algebras for oriented graphs

An oriented graph is a set of vertices with one-way directed edges between some of them. The theory of oriented graphs has developed enormously within the last decades. There is an extensive literature on the topic and many of these papers contain, not only interesting theoretical results, but also important algorithms as well as real-life applications.

The central problem in the area is the Caccetta-Häggkvist conjecture, which states that if from each vertex there are edges to at least $1/l$ fraction of the vertices, then there exists a directed cycle of length at most l. This problem was considered many times by a number of researchers in the last 40 years, but still remained open. Great significance of the Caccetta-Häggvist conjecture can be seen in particular in the fact that in 2006 a special workshop only about this conjecture was organized. In this research project, it is planned to solve some open problems highly related with the Caccetta-Häggkvist conjecture, which solution can help in proving this conjecture.

The first objective of the project is to solve the Kelly-Kühn-Osthus conjecture, which states, in the most interesting open case, that if for each vertex of an oriented graph there are edges to and from more than a quarter of the vertices, then the graph contains a directed cycle of length 6. Further research objectives include solving this conjecture without the assumption on edges coming from more than a quarter of the vertices (keeping only the assumption on edges going to more than a quarter), and proving its generalization for bigger directed cycles and other orientations of cycles. Solution of those problems can provide more tools of forcing cycles, that might be very useful for proving the Caccetta-Häggkvist conjecture.

Another project objective is to find the maximal possible number of directed cycles of length 4 (without any chords) in oriented graphs of a certain size. The believed extremal construction for this problem is also the believed extremal construction for the first and most considered open case of the Caccetta-Häggkvist conjecture, and so, developing techniques to prove this problem might be very helpful for proving the conjecture.

All of the above mentioned research tasks are planned to be achieved by use of new flag algebras techniques, which reduce the graph theory problem to the semi-definite programming, that can be solved numerically using a computer. Flag algebras methods were introduced only few years ago, but already have a significant impact on graph theory. By use of them many important problems, often open for few decades, were proven. It shows also, why it is so important to learn more about capabilities and limitations of those methods, and their further development.