## Educational project summary

Many systems of surrounding us real world can be characterised by a finite number of quantities $\boldsymbol{x} = (x_1, \ldots, x_n)$ describing uniquely their states. Model of a real system deduced from observations or theory tells us how quickly the system changes when it is at a state $\boldsymbol{x}$. In the mathematical form it takes the form of a system of differential equations $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$. Thus the behaviour in time of a real system described by a time evolution of its state $\boldsymbol{x}(t)$ is given in implicit way by differential equations.

Having a model the fundamental problem is to find a general solution for the corresponding system of differential equations, that is, to find the explicit form of dependence $\boldsymbol{x}(t)$. If we are able to do this, then we say the our system of differential equations is solved and we can forecast the behaviour in time of our real system. This is very important problem also for applications. One can imaging that after launching of a satellite we are every interested in tracking its motion.

Although the mathematical theory of ordinary differential equations is well-developed for majority of differential equations, which appear in different branches of science, one cannot hope to obtain its solutions in a explicit form. Fortunately, there exists an approach that enables to obtain rigorous information about the behaviour in time of our system and even in some cases also to find its solutions in an indirect way. To this aim we have to know certain functions $F(\boldsymbol{x})$ depending on the state of the system $\boldsymbol{x}$ which do not change during the time evolution of the system, i.e. $F(\boldsymbol{x}(t)) = F(\boldsymbol{x}(0))$ for an arbitrary $t$. We called them first integrals. If we are able to find a big enough number of independent first integrals, then system is called integrable and we can find its solutions. On the other hand, the absence of first integrals means that the behaviour of our system is complicated or chaotic. If we know only a few first integrals (less than necessary for integrability) we can reduce the dimension of the system that means reduce the number of quantities necessary for description of the system state.

For the most systems of physical origin the best known examples of first integrals are their energy, the linear momentum and the angular momentum and they are directly related to conservations laws of certain physical quantities. However, for many systems there exist first integrals that do not have such a nice physical interpretation and the question is how to find them. The first results concerning presence of first integrals and integrability one can find already in the famous Newton *Principia*. Henceforth the quest for first integrals has started and new integrable cases of important physical systems are considered as a big achievement of mathematics and physics. Let us mention here very non-trivial Kovalevskaya case in the dynamics of a rigid body honoured by Prix Bordin of the French Academy of Science. Integrable systems are rare but very important because give us complete information about physical systems and in fact most of our knowledge about real word is due to such systems. We just mention the integrable and solvable harmonic oscillator that is used as a model of various systems of the real world.

The direct method for searching of first integrals is based on a very simple idea. We assume that putative first integral has a specific form. For example, it is a polynomial with respect to certain variables and coefficients of this polynomial are just unspecified functions. From the fact that a first integral does not change during the evolution of the system one can obtain conditions for the unspecified coefficients. They have the form of other differential equations and in many cases these equations are simpler for solving than the original system. The method is very old and recently other methods of searching first integrals have appeared. But it has a big advantage in comparison with these modern approaches, namely as a starting point we take just our systems of differential equations without additional requirements. The main aim of this project is to develop theoretical methods and algorithms for improving the direct search method.

The second important aim of the project is to check efficiency of developed tools in applications for a systematic search for new integrable cases for certain classes of systems of differential equations. These classes were chosen on the one hand as important classes of physical systems: Hamiltonian systems with two degrees of freedom in curved spaces and Hamiltonian systems with algebraic potentials. On the other side they are the systems for that authors of the project very recently got very strong necessary integrability conditions. Systems that satisfy necessary integrability conditions are the best candidates for integrable systems but to confirm this it is necessary to give explicitly first integrals. This is exactly the contents of the third task of this project.

We believe that developed methods and algorithms as well as new integrable cases will be interesting for specialists in theory of dynamical systems and for scientists from various branches of science using in their research systems of differential equations.