DESCRIPTION FOR THE GENERAL PUBLIC

The project is concerned with research in the field of applied topology. Some of the proposed research topics constitute continuation of previous studies of investigators, but new directions are also included. More precisely, we have planned four distinct parts of research with time spent on each part corresponding proportionally to the order of proceeding presentation.

The first group is formed of topics related to the Bourgin–Yang theorem for various groups of symmetries. This theorem is a finer version of the classical Borsuk-Ulam theorem. The latter states (in its classical version) that there does not exist a continuous map $f\colon S(\mathbb{R}^n) \to S(\mathbb{R}^m)$ of spheres which satisfies $f(-x) = -f(x)$, provided that $n > m$. On the other hand, the Bourgin–Yang theorem (also in its classical statement) says that if $f\colon S(\mathbb{R}^n) \to \mathbb{R}^m$ is a map satisfying $f(-x) = -f(x)$, then $\dim f^{-1}(0) \geq n - m - 1$. In particular, if $n > m$ then the dimension of this set is ≥ 0 , thus it is nonempty, and the conclusion of the Borsuk–Uam theorem follows.

The Borsuk–Ulam theorem has many applications, for it is a fundamental tool in proofs of theorems about equipartition, both in discrete (the "necklace problem" of partitioning a necklace into pieces with the same number of beads of a given color by a minimal number of cuts) and continuous (the "sandwich problem" of partitioning a sandwich consisting of roll, ham and cheese into two parts with equal weight of all ingredients) cases. While the Borsuk–Ulam theorem ensures the existence of such partitions, the Bourgin–Yang theorem gives qualitative information about the size of the set of all possible partitions — it rises with the difference $n - m - 1$. E.g. if the sandwich in $\mathbb{R}^3$ consists of two ingredients, then this set is of dimension 1. If we admit more complicated groups of symmetries than $\mathbb{Z}_2 = \{\mathrm{Id}, -\mathrm{Id}\}$, and we wish to have more general spaces as the domain and co-domain of f, then mathematical setting becomes much more complicated for both theorems. However many situations in combinatorics require such generality. A far-reaching discussion of possible generalizations and applications of the Borsuk–Ulam theorem is included as one of research topics of the first part of this research proposal. It is also perhaps worth pointing out that the Borsuk–Ulam has applications outside of mathematics, e.g. in game theory.

The second thematic group will be concentrated on the study of topological complexity of a space X equipped with a finite number of symmetries. The theory of topological complexity, introduced at the beginning of this century, is now one of the most rapidly developing areas of applied topology. Roughly speaking, if the space of all possible states of a mechanical system (a robot) is described by a space X of topological complexity equal to n, then one requires to issue at least n commands to the robot in order to ensure that it will be able to motion plan on its own without the danger of failure of its operating system. In turns out that this number can be described very precisely in homotopy-theoretic terms by counting partial continuous sections of the path-space fibration $PX \to X \times X$. In the case when X has a finite symmetry group it is possible to define "topological complexity with symmetry" in many distinct ways, depending on how one interprets the model of a topological robot. This leads to different definitions (up to now there are four, two of them invented by investigators of this project). The study of these notions, a comparison of their properties, relations with the theory of transformation groups, and potentially introduction of yet another approach to the problem is the second aim of our project.

The third thematic group is about constructing and studying an invariant which we tentatively call a "linearization" of a homomorphism of groups, and, later on, its spectrum. Such a construction is already known for homomorphisms of torsion free nilpotent groups and their finite extensions, in which case the linearization is an integral matrix of finite size. It seems that this construction can be carried out for an essentially larger class of residually nilpotent groups. This time, however, the linearization would be an infinite product of integral matrices of finite size. In this general case this notion would most likely not have a topological interpretation, but could yield a new invariant in geometric group theory, assigning to homomorphism (which in general is a complicated object) a simpler object (an integral matrix). The study of properties of this invariant seems to be worthy of effort despite the fact that as of now the precise methodology of investigation in not really described.

The last part of our study is concerned with certain subspaces of functional spaces which can be defined thanks to the existence of a linear action of $O(N)$ in domains of functions, and whose elements have special geometrical properties, e.g. they change the sign and their sets of zeroes contain unions of hyperplanes. As a consequence, solutions of variational problems with $O(N)$-symmetry found in these subspaces will also have these properties. Describing their properties and counting those of them which are mutually perpendicular will automatically give information about the number of disjoint series of solutions with the mentioned properties.