Ergodic and Topological Properties of Dynamical Systems with Generalized Specification

A *dynamical system* is a mathematical model for a phenomenon that evolves over time. It consists of X, a phase space, and T — a transformation which defines the dynamics in this model by successive iteration. A T-invariant measure μ is called *ergodic*, if the dynamics is indecomposable into two sets on which the dynamics occurs independently with the requirement that both of these sets have positive μ-measure. The *topological entropy* says how complicated is the behavior of the dynamics defined by the map T. The *measure entropy* tells us how complex is the dynamical system with respect to the given measure. It follows from the *variational principle* that the topological entropy equals the supremum of the measure entropies over all ergodic measures. If the part of the space on which the dynamics is interesting has a large μ-measure, then the measure entropy of μ is not far from the topological entropy. Anatole Katok posed the following question: *what are the possibly weakest assumptions under which the variational principle is the only obstacle for the existence of an ergodic measure with the given entropy.* In other words, we would like to characterize the class of pairs (X, T) satisfying the following property: for every number between 0 and the topological entropy there exists an ergodic measure μ such that the measure entropy of μ is equal to t.

The family of all invariant measures has the structure of the *Choque simplex*, which in a way generalizes a triangle to the case of arbitrary (possibly infinite) dimension. The extreme points of this simplex (which one can imagine as vertices) are equal to the ergodic measures. It sometimes happens that these measures are dense in the whole simplex. This phenomenon is far from being intuitive and determines the simplex almost uniquely — there exists only one non-trivial simplex with this property. This object is called the Poulsen simplex and possesses many other remarkable features.

An *orbit* of a point is the sequence of all places in the space which will be visited by this point in a finite time. Imagine that for any tuple of points $(x_1, \ldots, x_n)$ which satisfies some technical assumptions one can find a point which tracks the orbit of x_1 during some time, after that changes to the orbit of x_2 and so on — up to x_n . The capability to glue pieces of orbits of different points is called a *specification-like* property and implies that the simplex of invariant measures is either a singleton or the Poulsen simplex.

The *entropy density* of ergodic measures means that every invariant measure can be approximated by ergodic measures in such a way that the entropy of this measure is also a limit of the entropies of measures estimating it. Pfister and Sullivan proved that if a system has some specification-like property (the so called almost specification property), then the ergodic measures are entropy dense. This is strictly connected to the Katok's problem. The entropy density of ergodic measures is only one of many remarkable consequences of satisfying a specification-like property.

Realizing the project I will focus on the following problems:

1. Extending the specification theory — looking for the new specification-like properties which are possessed by some smooth dynamical systems or $\mathbb{Z}^d$-actions; checking what are the dynamical consequences of them,
2. Constructing new examples of dynamical systems illustrating significance of the smoothness-type assumptions in the classical theorems concerning the outlined above theory. There are many phenomena which hold in a non-smooth setting, for example in symbolic dynamics, but remain unknown in the smooth case. This is the so called *problem of smooth models*. If one found the smooth and uniquely ergodic system with positive topological entropy, then the Katok's problem would be answered negatively.

Beside the above-mentioned problems I will also try to solve some other problems which are strictly connected with the main topic of the project.