Description for the general public

Gaussian random variables, i.e. vectors, which density is the Gaussian bell curve, are very important as the application in physics, statistics or economy. Usually one investigate not only one Gaussian ensemble, but many of them. Such a sequence of n Gaussian variables is called n-dimensional Gaussian vector. Gaussian vectors are very special; we can restore all the information about them by the covariance matrix, which is $n \times n$ array with the covariance of i-th and j-th coordinate of our vector in i-th column and j-th row. For example the coordinates of Gaussian random vector are independent if and only if they are uncorrelated, what means that the only non-zero terms in covariance matrix are on the diagonal. The other surprising property is the theorem, proven by Cramer, which says that if two independent copies X and Y of the same random variable (i.e. the independent ensembles of the same distribution) are such that $X - Y$ and $X + Y$ are independent, then X has to be the Gaussian random variable. Of course, the properties we have mentioned are typical for Gaussian vectors and fail for wider other classes of random vectors.

However, there are also surprising properties of Gaussian distribution, which holds true for some other distributions. They include the concentration of measure phenomenon. It turns out that Lipschitz functions (which are functions with at most linear growth) deviate from their mean with respect to the n-dimensional Gaussian measure (i.e. the distribution of Gaussian vector of independent coordinates of distribution $\mathcal{N}(0,1)$) less than t with high probability, exponentially close to one. We can think about it in the following way: in high dimension Lipschitz functions look like constant from the Gaussian measure point of view.

Another interesting property of Gaussian vectors is the behaviour of their moments. For every norm $\|x\|$ (namely a homogeneous, shift invariant distance) by the p-th moment we mean $\mathbb{E}\|G\|^p$, where G is is our n-dimensional standard Gaussian vector. The concentration property implies the exact behaviour of this moment with respect to p. It turns out that, up to a multiplicative constant independent on n and a norm, $(\mathbb{E}\|G\|^p)^{1/p}$ is equal to $\mathbb{E}\|G\| + \sigma(p, X)$. Here by $\sigma(p, X)$ we denote a weak moment with respect to our norm, which is often a quantity smaller than $(\mathbb{E}\|G\|^p)^{1/p}$.

Let us mention one more property of Gaussian vectors, called the Sudakov minoration. It says that the mean of the biggest coordinate of the standard Gaussian vector is equal, up to a multiplicative constant again, to a square root of logarithm of the dimension of the vector. More precisely, the Sudakov theorem allows to estimate the mean of the biggest coordinate of the arbitrary Gaussian vector, but under the square root of the logarithm a more complicated quantity (depending on the geometry of n-dimensional space with a structure coming from the covariance matrix) appears.

We have mentioned that the concentration property, the comparison of weak and strong moments and the Sudakov minoration hold for the wider class of random vectors than the Gaussian vectors only. In fact, every one of them is known to hold in some other special cases, but the conjectures that every one of them (separately) is valid for every logarithmically concave vector (which is a random vector with a density of a form $e^{-f(x_1,x_2,x_3,\ldots,x_{n-1},x_n)}$ and f is convex; for the Gaussian vector f is a symmetric quadratic polynomial) remains open. Our research project is dedicated to the investigation of this problem. We will tend to the solution of the conjectures for wider and wider classes of logarithmically concave vectors. The expected results will help to understand the geometric properties of the very important in probability theory class, namely the log-concave vectors, which are a generalization of Gaussian vectors. The proof of any of the conjectures in any unknown case will provide a new tool for proving other theorems, as the concentration property, the comparison of weak and strong moments and the Sudakow minoration imply nontrivial corollaries.