LÉVY PROCESSES WITH COMPLETELY MONOTONE JUMPS

DESCRIPTION FOR THE GENERAL PUBLIC

Introduction: Lévy processes. Consider the following simplistic model of the amount of water in a lake: The water leaves the dam in a continuous fashion at a constant rate v, and flows into the lake instantly when it rains. Suppose that each rain carries an equal amount of water Z, and that it comes at unpredictable, random times. Then the water level X_t at time t is described by the expression $X_t = X_0 + Z \times N_t - v \times t$, where N_t is the (random) number of rainfalls that occurred between time 0 and t.

The very same model appears in *queueing theory*: if v is the rate at which a task is processed and Z is the size of an individual task, then X_t describes the workload of the queue at time t. A reversed process is in turn the standard model in *risk theory*: if an insurance company collects premiums at a continuous rate v and Z is the size of a single claim, then company's surplus at time t is given by $X_t = X_0 - Z \times N_t + v \times t$.

In a slightly more realistic model, the rainfalls, tasks and claims are allowed to change over the time in a complete random fashion. In this case

$$X_t = x_0 + (Z_1 + Z_2 + ... + Z_{N_t}) - v \times t,$$

where v and random variables $Z_1, Z_2, \ldots$ can take both positive and negative values. Every process of this form is a *Lévy process*: a random (stochastic) process, whose increments are random variables that do not depend on the current time, the value of the process and its past behaviour. Moreover, in an appropriate sense, every Lévy process can be represented in a form very similar to the one discussed above.

A Lévy process is said to have *completely monotone jumps* if the distribution of random variables $Z_1, Z_2, \ldots$ is sufficiently regular; in particular the density function of this distribution needs to be an increasing and concave up function on the interval $(-\infty, 0)$, and a decreasing and concave up function on the interval $(0, \infty)$. This condition is satisfied, for example, in the Cramér–Lundber model in risk theory and in limit (fluid) models in queueing theory.

Project objectives. In each of the models described above, the instant at which X_t crosses the level 0 is critical: either the lake dries up, the queue becomes empty, or the company goes bankrupt. For this reason it is important to know what is the probability $P(t_0, x_0)$ of the event that X_t becomes negative at least once before time t_0. (For the lake or a queue, it is also reasonable to ask whether the process reached its *maximal* permissible level.) This problems lies at the heart of *fluctuation theory for Lévy processes*.

It may be surprising to learn that despite decades of development of this theory, a formula for $P(t_0, x_0)$ is known only for few special cases. One of the main goals of this research project is to derive an explicit formula (in an integral form) for $P(t_0, x_0)$ for Lévy processes with completely monotone jumps, as well as study the properties of this quantity for more general Lévy processes.

Other objectives are related to analytical properties of Lévy processes with completely monotone jumps and their multi-dimensional (vector-valued) generalisations.

Significance. This research project belongs to probability theory, the main subject of study is a class of random processes. However, because of numerous links with other areas of mathematics, the outcome of the project will have significant impact on the theory of partial differential equations and potential theory.

Selected results may also find practical applications. Evaluation of $P(t_0, x_0)$ in real-life models in queueing theory, risk theory or mathematical finance is typically time-consuming. Currently two methods are used: approximation with processes for which an explicit expression for $P(t_0, x_0)$ is known, and Monte Carlo simulations (sampling numerous realisations of the process). This research project aims at a new formula for $P(t_0, x_0)$, valid for a much wider class of Lévy processes. In some applications this may simplify the numerical evaluation of this quantity.