DISCRETE HARMONIC ANALYSIS

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Our research plans will be concentrated on the following problems:

(1) ℓ^p-estimates for maximal functions corresponding to discrete multi-parameter Radon averaging operators.
(2) ℓ^p-estimates for maximal functions corresponding to discrete multi-parameter truncated singular Radon transforms.
(3) ℓ^p-theory for discrete non-commutative operators of Radon type: maximal and variational estimates.

The motivations to carry out this research in these directions are threefold. Firstly, the field of discrete analogues in harmonic analysis is still relatively young, and despite the fact that much progress has been made recently, many interesting and important problems remain open. The problems we are going to investigate will fall into two major streams within the theory of discrete analogues. On the one hand, we will be concerned with the discrete multi-parameter theory for operators of Radon type defined along polynomial surfaces. On the other hand, the theory of one-parameter, non-commutative discrete operators of Radon type modeled on polynomial mappings is still in its infancy an we plan to make substantial progress in this direction. All the questions we intend to investigate arose in ongoing collaboration of the author with Professor James Wright and Professor Elias M. Stein and recently also during discussions with Professor Alexandru Ionescu.

Secondly, the questions raised in this project find applications in ergodic theory. The usefulness and importance of ergodic theory in other branches of mathematics (probability theory, additive combinatorics, number theory, dynamical systems) and beyond (classical or statistical mechanics) is invaluable. Recently, Green and Tao, using tools from ergodic theory, established that the set of prime numbers inherits 'additive structure' from the set of integers by proving the existence of arbitrarily long arithmetic progressions in the primes. Ergodic questions concerning pointwise convergence or convergence in mean can be transferred into questions in discrete harmonic analysis. This aspect is especially important to us. Problems related to pointwise convergence require the most sophisticated tools developed to date in harmonic analysis. Among these tools are maximal functions, oscillation or r-variatinal seminorms. All these objects will be studied in the context of our questions. Further, developments in this area has great importance for harmonic analysis.

Thirdly, another reason which makes discrete analogues so fascinating is the fact (as recent results show) that the phenomena occurring there may completely differ from what happens when their continuous analogues are considered. These circumstances make discrete analogues very challenging since intuition from the continuous harmonic analysis often does not hold. Therefore, it is tempting to study discrete analogues since new methods are necessary.

The project has an interdisciplinary nature, since many interesting questions arise at the interface of harmonic analysis, number theory and ergodic theory when discrete averaging operators or singular integrals modeled on polynomial mappings are studied. So far we have mentioned only their applications in ergodic theory. But our problems, due to their arithmetic flavor, have deep connections with analytic number theory via exponential sums. Exponential sums naturally come up in these discrete analogues when we use Fourier transform methods to analyze multipliers corresponding to our operators.

Exponential sums of large degree play a decisive role in the analysis of problems spanning the analytic theory of numbers, and consequently the estimation of their mean values is of central significance. In the 1930's Vinogradov made a breakthrough obtaining new estimates for such mean values by exploiting the translation-dilation invariance of associated systems of Diophantine equations. He was able to obtain robust new estimates for exponential sums going well beyond those made available via the differencing or TT^* methods of Weyl and van der Corput. Decisive progress followed in such topics as Waring's problem, the zero-free region for the Riemann zeta function, the distribution modulo 1 of polynomial sequences and discrete restriction problems.

Recently, estimates of exponential sums turned out to be critical in Green's proof of Roth's theorem in the primes concerning the existence of arithmetic progressions of length at least three in subsets of the prime numbers with non-vanishing relative upper density. Finally, we emphasize that recent developments in the theory of exponential sums resulted in the full solution of the ternary Goldbach conjecture, which asserts that every odd integer $N > 5$ is a sum of three primes. Helfgott completed the verification that the ternary Goldbach conjecture is true.

In view of these recent achievements we can fully acknowledge the usefulness of exponential sums. However, in the context of discrete harmonic analysis exponential sums still require better understanding. Therefore, deep studies are necessary. We hope that our research plans will be important in this aspect as well.