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Differential equations model various phenomena in physics, chemistry, biology, meteorology, astronomy. One of the fundamental models is heat equation. Imagine an infinite rod isolated from the environment, so no heat escapes from the rod. If in a fixed time the distribution of the temperature in the rod is given by some function $f(x)$, then as time goes by, the temperature will equalize. This process may be described by the adequate differential equation. If we denote by $u(t,x)$ the temperature of the rod at time t in the point x then the function u satisfies heat equation with initial condition $u(0,x) = f(x)$, saying that at the beginning of the process temperature was given by f. The function $u(t,x)$ may be expressed by f and Gauss-Weierstrass kernel, which plot is the well known Gaussian curve.

This model may and even must be complicated because in fact heat propagates in various bounded objects (also two and three dimensional). While modeling this process one has to take into account the shape of the domain, when the heat propagates and the way the heat radiates into the environment. Of course, one has to take under consideration the material of which the studied object is made of. A wooden stick and a metal rod heat up in a different way. The solution of the heat equation with the given condition on the boundary may be expressed by the function called Dirichlet heat kernel.

If, in a perfect vacuum, there is a positive charge at one point x_0, it would generate a potential field, which can be described by the so-called Newton's potential equal to $|x-x_0|^{-1}$. However, if the charge would be in the middle of a certain area, such as a ball, whose edge would be perfectly isolated from the environment, then, inside the ball the potential would be given by another function, called the Green function. This function is also the solution of a certain differential equation with the appropriate boundary condition.

There is a close relationship between the Dirichlet kernel and the Green function. This relationship, apart from an analytic connections, has also a probabilistic background. If we observe a molecule in a liquid, then, as a result of collisions with other particles, it will be moving chaotically. This fact was observed for the first time in 1827 by the Scottish biologist Robert Brown, and the motion of the molecule was called Brownian motion. Mathematically, this phenomenon is described in the language of probability by using the theory of random processes. Since the particle moves randomly, one can only determine the probability that at any given moment this particle will be in a certain place. If we observe the movement of the molecule in a certain area until hitting the edge of this area, a function describing the probability of the position of the particle is the Dirichlet heat kernel discussed above. In contrast, the Green function is related to the length of the time that the molecule spends in a certain place.

Of course, there are many generalizations of these models. In the heat equation there is the second derivative, in higher dimension, Laplace operator. The researchers consider other operators, in particular, the models involving fractional Laplacian are of the great interest. Another processes are also studied, both the ones moving continuously and jumping processes. In particular, it is worth mentioning stable processes, which has the same relationship with fractional Laplacian as the Brownian motion with the Laplace operator.

Analyzing these examples, one can see that the theories of differential equations and probability intertwine. The same objects are present in both fields but describing different phenomena. Therefore, it is not surprising that some studies are carried out at the junction of these two branches of science, and so is the case of this project. Our research will address both of these theories, but with greater accent on differential equations. We will study the solutions of equations, in some way, similar to the heat equation. One may consider them as the perturbed heat equation, where, due to some additional factors, an evolution process has a different nature and is described by another equation, often non-linear. Our goal is to take a look how similar are solutions of the perturbed equations to the solution of the heat equation. Sometimes it happens that the solution, after the perturbation of the equation, behaves differently, and such cases are particularly interesting. We will also study the behavior of the Green function and Dirichlet heat kernel for a certain class of jumping processes, which also includes stable processes.