1 Introduction

We would like to solve combinatorial problems emerging from real-life applications fast. However, for many problems we do not know efficient algorithms. We would like to know if it is because we have not developed a required approach or tools yet or perhaps it is impossible at all. In the latter case it suggests to reformulate our statement of the problem. The knowledge *why* the problem is hard may help us to find a suitable restatement of the problem which is actually efficiently solvable (e.g. approximation, FPT and kernelization, smoothed analysis, noisy input model etc.).

On the other hand if we already know efficient algorithms for the problem, we would like to know whether they are optimal. If not, the knowledge what complexity we should expect can be a valuable hint during the design of a new algorithm.

Our goal is to discover the limitations of the current algorithm theory and the current problems.

2 Research Project Objectives

In particular under strong complexity assumptions we would like to prove tight lower bounds for the complexities of well known problems.

2.1 Edge Coloring

The problem of Edge Coloring or in other words the problem of dividing the set of the edges of a graph into matchings was studied about one hundred years ago by Kőnig. This resulted in the famous Kőnig's Theorem from 1916. Another important results are a Shannon's Theorem for multigraphs from 1949 [11] and Vizing's Theorem for simple graphs from 1964 [13]. The NP-hardness of Edge Coloring was an open problem for some time and was finally proven in 1981 by Holyer [6] by producing a 3-regular instance and in 1983 by Leven and Galil [8] by producing a k-regular instance for any $k \geq 3$. The Edge Coloring problem was studied in a great number of publications and in many different settings. The best exact algorithm in the general case works in $O^*(2^m)$ time [1].

2.2 Set Cover

The Set Cover problem is one of the original Karp's 21 NP-complete problems from 1972 [7]. It is equivalent to Hitting Set and captures many problems like Vertex Cover, Dominating Set, Set Packing and others. The Set Cover problem has a long story of lower bounds for the approximation ratio and in 2013 Dinur and Steurer showed that it cannot be approximated with a factor $(1 - o(1)) \cdot \ln n$ under the assumption of $P \neq NP$ [3]. This bound is tight due to the greedy algorithm [2]. The problem is also $W[2]$-complete. Although we know that the problem is very hard still we do not know if it can be solved in $O^*(2^{\varepsilon n})$ for any $\varepsilon < 1$. Such a bound would be tight because of a dynamic programming algorithm from 2004 [4].

2.3 *a*:*b*-Coloring

The a:b-Coloring problem was introduced in 1976 by Stahl [12] as a continuation of the line of research from [5]. The problem is known under many names: a:b-Coloring, b-Tuple Coloring, b-Fold Coloring, Multiple Coloring and was considered in many publications. The Multi-Coloring problem is a variant of the problem such that the number of required colors is given for each vertex separately. The exact complexity of the a:b-Coloring problem is really intriguing because on one hand it is a generalization of the classical Graph Coloring problem, but on the other hand it seems to be very close to the linear programming relaxation of the Graph Coloring problem (Fractional Chromatic Number) [9,10].

2.4 Further Goals

Very recently there were quite a few results on showing lower bounds for polynomially solvable problems like Longest Common Sequence, Edit Distance, Local String Alignment, Dynamic Strong Connected Components, Dynamic Single Source Reachability, Dynamic Graph Diameter, Graph Diameter (for approximation, in contrary to our goal) and Orthogonal Vectors. All of these results evolved as consequences of the study on the complexity of the NP-hard problems. They are using the same methodology (reductions from SAT, sometimes indirectly) as we do in our research. Therefore we plan to study polynomially solvable problems like: Triangle Detection, All Pairs Shortest Paths, Graph Diameter, 3SUM and Optimum Binary Search Tree in order to found their lower bounds.

3 Implications

The practical application of this knowledge is straightforward. For particular formulations of the real-life problems we know what complexity we should expect. We know then if there are chances for the existence of the faster algorithms, or we know that in order to solve our practical problem faster we need to reformulate the statement. Therefore such results are very important also from the practical point of view.

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